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On the Derivative of a Polynomial

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Abstract: For an arbitrary polynomial $P(z)$, let $M(P,r) = \max_{|z|=r} |P(z)|$ and $m(P,r) = \min_{|z|=r} |P(z)|$, $(r > 0)$. For a polynomial $p(z) = \sum_{j=0}^{n} a_j z^j = a_n \prod_{\nu=1}^{n}(z - z_\nu)$, of degree n, having all its zeros in $|z| \leq k, (k \geq 1)$, with a zero of order $s, (s \geq 0)$, at 0 and

$F_0, F_1, F_2, G_{n-s}, F_3, F_4, H_{n-s}, F_{n-s}, B_0, B_1, E_{n-1}, B_2, B_3, D_{n-1}$ and B_{n-1},

as in Theorem, we have obtained a refinement

$$\begin{aligned} M(p^{'},1) \geq & \frac{2}{1+k^{n-s}}(\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|})M(p,1) \\ & + \frac{k^{n-s}-1}{k^n(1+k^{n-s})}(\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|})m(p,k) \\ & + \frac{2}{k^{n-s}(1+k^{n-s})}(\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|})F_{n-s} + \frac{B_{n-1}}{k^{n-1}}, \end{aligned}$$

of our old result (1997), thereby obtaining a new refinement of known results

$$M(p',1) \geq \frac{n}{1+k^n}M(p,1), (1973)$$

and

$$M(p',1) \geq \frac{2}{1+k^n}(\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|})M(p,1), (1983).$$

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1. Introduction and statement of result

For an arbitrary polynomial $P(z)$, let $M(P,r) = \max_{|z|=r} |P(z)|$ and $m(P,r) = \min_{|z|=r} |P(z)|, (r > 0)$. For a given polynomial $p(z)$, concerning the estimate of $|p'(z)|$ on $|z| \leq 1$, we have the following well-known result due to Turán [9], suggesting a lower bound for $M(p',1)$.

Theorem A. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq 1$ then*

$$M(p',1) \geq \frac{n}{2} M(p,1).$$

The result is sharp with equality for the polynomial $p(z)$ having all its zeros on $|z| = 1$.

Malik [8] obtained a generalization of Theorem A, namely

Theorem B. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq k, (k \leq 1)$ then*

$$M(p',1) \geq \frac{n}{1+k} M(p,1).$$

The result is sharp with equality for the polynomial $p(z) = (z+k)^n$,

and Govil [4] obtained the generalization

Theorem C. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq k, (k \geq 1)$ then*

$$M(p',1) \geq \frac{n}{1+k^n} M(p,1).$$

The result is sharp with equality for the polynomial $p(z) = z^n + k^n$.

Aziz [1] obtained a refinement of Theorem C in the form

Theorem D. *If all the zeros of the polynomial $p(z) = a_n \prod_{j=1}^{n} (z - z_j)$, of degree n lie in $|z| \leq k, (k \geq 1)$ then*

$$M(p',1) \geq \frac{2}{1+k^n} \Big(\sum_{j=1}^{n} \frac{k}{k + |z_j|} \Big) M(p,1).$$

The result is best possible with equality for the polynomial $p(z) = z^n + k^n$,

which was further refined by Govil [5] to give

Theorem E. *Let $p(z) = \sum_{j=0}^{n} a_j z^j = a_n \prod_{t=1}^{n} (z - z_t)$, be a polynomial of degree $n \geq 2, |z_t| \leq K_t$,*
$1 \leq t \leq n$ and let $K = \max(K_1, K_2, \ldots, K_n) \geq 1$. Then

$$\begin{aligned} M(p',1) \quad &\geq \quad \frac{2}{1+K^n} \Big(\sum_{t=1}^{n} \frac{K}{K+K_t} \Big) M(p,1) + \\ &\frac{2|a_{n-1}|}{1+K^n} \Big(\sum_{t=1}^{n} \frac{1}{K+K_t} \Big) \Big(\frac{K^n - 1}{n} - \frac{K^{n-2} - 1}{n-2} \Big) + |a_1| \Big(1 - \frac{1}{K^2} \Big), n > 2 \end{aligned}$$

and

$$\begin{aligned} M(p',1) &\geq \frac{2}{1+K^n}\Big(\sum_{t=1}^{n}\frac{K}{K+K_t}\Big)M(p,1)+\frac{(K-1)^n}{1+K^n}|a_1|\Big(\sum_{t=1}^{n}\frac{1}{K+K_t}\Big) \\ &\quad +|a_1|\Big(1-\frac{1}{K}\Big),\ n=2. \end{aligned}$$

The result is best possible with equality for the polynomial $p(z)=z^n+K^n$.

We, in our old result [6], had considered the polynomial having all its zeros in $|z|\leq k, (k\geq 1)$, with a possible zero of order $m, (m\geq 0)$, at 0 and had obtained the following refinement of both Theorem C and Theorem D.

Theorem F. *Let* $p(z)=\sum_{s=0}^{n}a_sz^s=a_n\prod_{\gamma=1}^{n}(z-z_\gamma)$ *be a polynomial of degree* n, *having all its zeros in* $|z|\leq k, (k\geq 1)$. *Then*

$$M(p',1)\geq \frac{2}{1+k^{n-m}}\Big(\sum_{\gamma=1}^{n}\frac{k}{k+|z_\gamma|}\Big)M(p,1)+\frac{C}{k(1+k^{n-m})}\Big(\sum_{\gamma=1}^{n}\frac{1}{k+|z_\gamma|}\Big)+D, \tag{1.1}$$

where

$$p(z)=z^mp_1(z),\ \text{with } p_1(0)\neq 0,\ \text{for some non-negative integer } m,$$

non-negative real number

$$C=\begin{cases} 4|a_{n-2}|\{c_{n-m-2}(k)-c_{n-m-4}(k)- \\ \Big(\frac{k^{n-m-1}-1}{n-m-1}-\frac{k^{n-m-3}-1}{n-m-3}\Big)\Big\} & ,n>4\ \&\ 0\leq m<n-4, \\ 4|a_{n-2}|\Big\{D_k-\Big(\frac{k^3-1}{3}-\frac{k^2-1}{2}\Big)\Big\} & ,n\geq 4\ \&\ m=n-4, \\ 4|a_{n-2}|\Big\{F_k-\frac{k^2-1}{2}\Big\} & ,n\geq 3\ \&\ m=n-3, \\ |a_{n-1}|k(k-1)^2 & ,n>2\ \&\ m=n-2, \\ (|a_n|k-|a_{n-1}|)k(k-1) & ,n\geq 1\ \&\ m=n-1, \\ 0 & ,n\geq 1\ \&\ m=n, \end{cases}$$

non-negative real number

$$D=\begin{cases} 2|a_2|\Big(\frac{1}{k}-\frac{1}{k^3}\Big)(\sqrt{k^2+1}-1) & ,n>4\ \&\ m\leq n-1, \\ 2|a_2|\Big(\frac{1}{k}-\frac{1}{k^2}\Big)(\sqrt{k^2+k+1}-1) & ,n=4\ \&\ m\leq n-1, \\ \frac{2|a_2|}{k}\Big(\sqrt{\frac{k^2+1}{2}}-1\Big) & ,n=3\ \&\ m\leq n-1, \\ |a_1|\Big(1-\frac{1}{k}\Big) & ,n=2\ \&\ 0<m\leq n-1, \\ 0 & ,n>1\ \&\ m=n, \\ 0 & ,n=1, \end{cases}$$

$$\begin{aligned} c_t(k) &= \int_1^k r^t\sqrt{r^2+1}dr, t>0,\\ D_k &= \int_1^k (r^2-r)\sqrt{r^2+r+1}dr \end{aligned}$$

and

$$F_k = \int_1^k r\sqrt{\frac{r^2+1}{2}}dr.$$

In (1.1) *equality holds for the polynomial* $p(z) = z^n + k^n$.

In this paper we have obtained a refinement of our old result, namely Theorem F, thereby obtaining a new refinement of Theorem C and Theorem D. More precisely we have proved

Theorem. *Let* $p(z) = \sum_{j=0}^n a_j z^j = a_n \prod_{\nu=1}^n (z - z_\nu)$ *be a polynomial of degree* n*, having all its zeros in* $|z| \le k, (k \ge 1)$*, with a zero of order* $s, (s \ge 0)$*, at* 0*. Then*

$$\begin{aligned} M(p',1) \quad &\ge \quad \frac{2}{1+k^{n-s}}\Big(\sum_{\nu=1}^n \frac{k}{k+|z_\nu|}\Big)M(p,1) + \frac{k^{n-s}-1}{k^n(1+k^{n-s})}\Big(\sum_{\nu=1}^n \frac{k}{k+|z_\nu|}\Big)m(p,k)\\ &\quad + \frac{2}{k^{n-s}(k^{n-s}+1)}\Big(\sum_{\nu=1}^n \frac{k}{k+|z_\nu|}\Big)F_{n-s} + \frac{B_{n-1}}{k^{n-1}}, \qquad (1.2) \end{aligned}$$

where

$$\begin{aligned} B_0 &= 0,\\ B_1 &= (k-1)|a_1|,\\ B_2 &= \max\Big(E_2|a_1|, 2|a_2|k\Big(\sqrt{\frac{k^2+1}{2}}-1\Big)\Big),\\ B_3 &= \max\Big(E_3|a_1|, 2|a_2|(k^2-k)(\sqrt{k^2+k+1}-1)\Big),\\ B_{n-1} &= \max\Big(E_{n-1}|a_1|, 2|a_2|D_{n-1}\Big), n-1 \ge 4,\\ E_{n-1} &= k^{n-1} - k^{n-3}, n-1 \ge 2,\\ D_{n-1} &= \Big(k^{n-2} - k^{n-4}\Big)\Big(\sqrt{k^2+1}-1\Big), n-1 \ge 4,\\ F_0 &= 0,\\ F_1 &= 0,\\ F_2 &= |a_{n-1}|k\frac{(k-1)^2}{2}, \end{aligned}$$

$$
\begin{aligned}
F_3 &= \max\Big(k^2|a_{n-1}|G_3, 2k|a_{n-2}|\Big(\int_1^k r\sqrt{\frac{r^2+1}{2}}dr - \frac{k^2-1}{2}\Big)\Big),\\
F_4 &= \max\Big(k^3|a_{n-1}|G_4, 2k^2|a_{n-2}|\Big(\int_1^k (r^2-r)\sqrt{r^2+r+1}dr - \Big(\frac{k^3-1}{3} - \frac{k^2-1}{2}\Big)\Big)\Big),\\
F_{n-s} &= \max\Big(k^{n-s-1}|a_{n-1}|G_{n-s}, 2k^{n-s-2}|a_{n-2}|H_{n-s}\Big), \quad n-s \geq 5,\\
G_{n-s} &= \frac{k^{n-s}-1}{n-s} - \frac{k^{n-s-2}-1}{n-s-2}, \quad n-s \geq 3
\end{aligned}
$$

and

$$
H_{n-s} = \int_1^k r^{n-s-2}\sqrt{r^2+1}dr - \int_1^k r^{n-s-4}\sqrt{r^2+1}dr - \Big(\frac{k^{n-s-1}-1}{n-s-1} - \frac{k^{n-s-3}-1}{n-s-3}\Big), \quad n-s \geq 5.
$$

In (1.2) equality holds for the polynomial $p(z) = z^n + k^n$.

Remark 1. In many cases, our Theorem gives a better lower bound for $M(p',1)$ than those given by other known results, as for the polynomial $p(z) = z(z^3+8)(z+3)$, having all its zeros in $|z| \leq 3$, we get

$$
\begin{aligned}
M(p',1) &\geq 25.5, \text{ by Theorem,}\\
M(p',1) &\geq 13.1, \text{ by Theorem F,}\\
M(p',1) &\geq 23.4, \text{ by Theorem E}
\end{aligned}
$$

and

$$
M(p',1) \geq 5.8, \text{by result [7, Theorem 1.7].}
$$

2. Lemmas

For the proof of Theorem we require the following lemmas.

Lemma 1. *If $p(z)$ is a polynomial of degree $n(\geq 2)$ then for all $R > 1$*

$$
M(p,R) \leq R^n M(p,1) - (R^n - R^{n-2})|p(0)|.
$$

Lemma 1 is due to Frappier et al. [3, Theorem 2].

Lemma 2. *Let $p(z)$ be a polynomial of degree $n(\geq 2)$ and let $R \geq 1$. Then*

$$
\begin{aligned}
M(p,R) &\leq R^n M(p,1) - |p'(0)|(R^{n-1} - R^{n-3})(\sqrt{R^2+1} - 1), \ n \geq 4,\\
M(p,R) &\leq R^n M(p,1) - |p'(0)|(R^2 - R)(\sqrt{R^2+R+1} - 1), \ n = 3
\end{aligned}
$$

and

$$
M(p,R) \leq R^n M(p,1) - |p'(0)|R\Big(\sqrt{\frac{R^2+1}{2}} - 1\Big), \ n = 2.
$$

Lemma 2 is due to Frappier et al. [3, Theorem 4].
Using Lemma 1 and Lemma 2 one easily obtains

Lemma 3. *If $p(z)$ is a polynomial of degree n then for $R \geq 1$*

$$M(p,R) \leq R^n M(p,1) - B_n(p,R),$$

where

$$\begin{aligned} B_1(p,R) &= (R-1)|p(0)|, \\ B_2(p,R) &= \max\Big(E_2(R)|p(0)|, R\Big(\sqrt{\frac{R^2+1}{2}}-1\Big)|p'(0)|\Big), \\ B_3(p,R) &= \max\Big(E_3(R)|p(0)|, (R^2-R)\Big(\sqrt{R^2+R+1}-1\Big)|p'(0)|\Big), \\ B_n(p,R) &= \max(E_n(R)|p(0)|, D_n(R)|p'(0)|),\ n \geq 4, \\ E_n(R) &= R^n - R^{n-2},\ n \geq 2 \end{aligned}$$

and

$$D_n(R) = (R^{n-1} - R^{n-3})(\sqrt{R^2+1}-1),\ n \geq 4.$$

Remark 2. One can note that Lemma 3 is trivially true for $n = 0$, with $B_0(p,R) = 0$.

Lemma 4. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < 1$ then*

$$M(p',1) \leq \frac{n}{2}\{M(p,1) - m(p,1)\}. \tag{2.1}$$

There is equality in (2.1) for $p(z) = \alpha + \beta z^n, |\alpha| = |\beta|$.

Lemma 4 is due to Aziz and Dawood [2].

Lemma 5. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n > 2$, having no zeros in $|z| < 1$ then for $R \geq 1$*

$$M(p,R) \leq \frac{R^n+1}{2}M(p,1) - m(p,1)\frac{R^n-1}{2} - |a_1|\Big(\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\Big). \tag{2.2}$$

Equality holds in (2.2) for $p(z) = z^n + 1$.

Proof of Lemma 5. It is similar to the proof of Lemma 4 [5] with one change: Lemma 4 instead of Lemma 2 [5].

Lemma 6. *If $p(z)$ is a polynomial of degree $n > 4$, having no zeros in $|z| < 1$ then for $R \geq 1$*

$$\begin{aligned} M(p,R) \leq\ & \frac{R^n+1}{2}M(p,1) - \frac{R^n-1}{2}m(p,1) - |p''(0)|\Big\{c_{n-2}(R) - c_{n-4}(R) \\ & -\Big(\frac{R^{n-1}-1}{n-1} - \frac{R^{n-3}-1}{n-3}\Big)\Big\}, \end{aligned} \tag{2.3}$$

where

$$c_t(R) = \int_1^R r^t\sqrt{r^2+1}dr, t > 0.$$

There is equality in (2.3) *for* $p(z) = \alpha + \beta z^n$, $|\alpha| = |\beta|$.

Proof of Lemma 6. It is similar to the proof of Lemma 4 [6] with one change: Lemma 4 instead of Lemma 2 [6].

Lemma 7. *If* $p(z)$ *is a polynomial of degree* $n = 4$, *having no zeros in* $|z| < 1$ *then for* $R \geq 1$

$$M(p,R) \leq \frac{R^n+1}{2}M(p,1) - \frac{R^n-1}{2}m(p,1) - |p''(0)|\left\{D_R - \Big(\frac{R^3-1}{3} - \frac{R^2-1}{2}\Big)\right\}, \tag{2.4}$$

where

$$D_R = \int_1^R (r^2-r)\sqrt{r^2+r+1}dr.$$

There is equality in (2.4) *for* $p(z) = \alpha + \beta z^n, |\alpha| = |\beta|$.

Proof of Lemma 7. It is similar to the proof of Lemma 5 [6] with one change: Lemma 4 instead of Lemma 2 [6].

Lemma 8. *If* $p(z)$ *is a polynomial of degree* $n = 3$, *having no zeros in* $|z| < 1$ *then for* $R \geq 1$

$$M(p,R) \leq \frac{R^n+1}{2}M(p,1) - \frac{R^n-1}{2}m(p,1) - |p''(0)|\Big(F_R - \frac{R^2-1}{2}\Big), \tag{2.5}$$

where

$$F_R = \int_1^R r\sqrt{\frac{r^2+1}{2}}dr.$$

There is equality in (2.5) *for* $p(z) = \alpha + \beta z^n, |\alpha| = |\beta|$.

Proof of Lemma 8. It is similar to the proof of Lemma 6 [6] with one change: Lemma 4 instead of Lemma 2 [6].

Lemma 9. *If* $p(z)$ *is a polynomial of degree* $n = 2$, *having no zeros in* $|z| < 1$ *then for* $R \geq 1$

$$M(p,R) \leq \frac{R^n+1}{2}M(p,1) - \frac{R^n-1}{2}m(p,1) - |p'(0)|\frac{(R-1)^2}{2}. \tag{2.6}$$

There is equality in (2.6) *for* $p(z) = \alpha + \beta z^n, |\alpha| = |\beta|$.

Proof of Lemma 9. It is similar to the proof of Lemma 8 [6] with one change: Lemma 4 instead of Lemma 2 [6].

Using Lemma 5, Lemma 6, Lemma 7, Lemma 8 and Lemma 9 one easily obtains

Lemma 10. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < 1$ then for $R \geq 1$*

$$M(p,R) \leq \frac{R^n+1}{2}M(p,1) - \frac{R^n-1}{2}m(p,1) - F_n(p,R), \tag{2.7}$$

where

$$F_1(p,R) = 0,$$

$$F_2(p,R) = |p'(0)|\frac{(R-1)^2}{2},$$

$$F_3(p,R) = \max\Big(G_3(R)|p'(0)|, \Big(\int_1^R r\sqrt{\frac{r^2+1}{2}}dr - \frac{R^2-1}{2}\Big)|p''(0)|\Big),$$

$$F_4(p,R) = \max\Big(G_4(R)|p'(0)|, \Big(\int_1^R (r^2-r)\sqrt{r^2+r+1}dr - \Big(\frac{R^3-1}{3} - \frac{R^2-1}{2}\Big)\Big)|p''(0)|\Big),$$

$$F_n(p,R) = \max\Big(G_n(R)|p'(0)|, H_n(R)|p''(0)|\Big), \ n \geq 5,$$

$$G_n(R) = \Big(\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\Big), \ n \geq 3$$

and

$$H_n(R) = \int_1^R r^{n-2}\sqrt{r^2+1}dr - \int_1^R r^{n-4}\sqrt{r^2+1}dr - \Big(\frac{R^{n-1}-1}{n-1} - \frac{R^{n-3}-1}{n-3}\Big), \ n \geq 5.$$

There is equality in (2.7) *for* $p(z) = \alpha + \beta z^n, \ |\alpha| = |\beta|$.

Remark 3. One can note that Lemma 10 is trivially true for $n = 0$, with $F_0(p,R) = 0$.

3. Proof of Theorem

It is similar to the main part of Proof of Theorem [6] with two changes:
Lemma 3 along with Remark 2 instead of Lemma 3 [6],
Lemma 10 along with Remark 3 instead of Lemma 4 [6].

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