Journal of
Mathematics
and Applications
JMA No 46, pp 149-162 (2023)

Existence of Traveling Profiles Solutions to Porous Medium Equation

Yacine Arioua and Nouredine Benhamidouche

Abstract: In this paper, we shall study the existence and uniqueness of solutions called "traveling profiles solutions" to the porous medium equation in one dimension. By these solutions, we generalize the results obtained by Gilding and Peletier who proved the existence of self similar solutions of type I, II and III to the same equation. The principal idea of our work is to convert the porous media equation in to an equivalent nonlinear differential equation, and to prove the existence and uniqueness of these new solutions under certain conditions.

AMS Subject Classification: 35C06.
Keywords and Phrases: Porous medium equation; Exact solution; Traveling profile solutions.

1. Introduction:

Consider the one dimensionel porous media equation, which is written as:

$$\left\{ \frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} (u^m) \right\}, \tag{1.1}$$

where $u > 0$, $x \in \mathbb{R}$, $t > 0$, and $m > 1$, is a fixed real number. Equation (1.1) is parabolic at any point (x,t) at which $u > 0$. However, at points where $u = 0$, it is degenerate parabolic. Equation (1.1) arises in many other applications, e.g, in the theory of ionized gases at high temperature [21] for values of $m > 1$, and in various models in plasma physics [5] for values of $m < 1$. Of course, for $m = 1$, equation (1.1) is the classical equation of heat conduction. In this paper we will focus on the case where $m > 1$.

Classes of weak solutions for the Cauchy problem and the Cauchy-Dirichlet problem of Eq. (1.1) were introduced by Oleinik, Kalashnikov and Yui-Lin [15], they proved existence and uniqueness of such solutions and they showed that if at some instant t_0, a weak solution $U(x,t_0)$ has compact support, then $u(x,t)$ has also compact support for any $t \geqslant t_0$. Gilding and Peletier [12], has study a class of similarity solutions of (1.1) for $0 < x < \infty$ and $0 < t \leqslant T$, where T is some positive constant. These solutions has three following form:

$$\begin{aligned} 1.\ u_1(x,t) &= (t+\tau)^{\alpha} f_1(\eta),\ \eta = x(t+\tau)^{\frac{-1}{2}(1+(m-1)\alpha)},\ \text{for } \tau > 0, && (1.2)\\ 2.\ u_2(x,t) &= (t-\tau)^{\alpha} f_2(\eta),\ \eta = x(t-\tau)^{\frac{-1}{2}(1+(m-1)\alpha)},\ \text{for } \tau > T, && (1.3)\\ 3.\ u_3(x,t) &= e^{\alpha(t+\tau)} f_3(\eta),\ \eta = x\exp(\frac{-1}{2}\alpha\,(m-1)\,(t+\tau)),\ \text{for any } \tau. && (1.4)\end{aligned}$$

After substitution of u_1, u_2 and u_3 into (1.1), they have obtained the following equations for the functions f_1, f_2 and f_3:

$$\begin{aligned} I.\quad (f_1^m)'' &= \alpha f_1 - \frac{1}{2}\{1+(m-1)\alpha\}\,\eta f_1',\quad 0<\eta<\infty, && (1.5)\\ II.\quad (f_2^m)'' &= -\alpha f_2 + \frac{1}{2}(\{1+(m-1)\alpha\}\,\eta f_2',\quad 0<\eta<\infty, && (1.6)\\ III.\quad (f_3^m)'' &= \alpha f_3 - \frac{1}{2}\alpha(m-1)\eta f_3',\quad 0<\eta<\infty. && (1.7)\end{aligned}$$

At the boundaries, the following conditions are imposed:

$$f_i\,(0) = U\ \ (\geq 0)\,,\ f_i\,(\infty) = 0,\ i = 1,2,3.$$

Thus the solutions $u_i(x,t)$ satisfy the lateral boundary conditions

$$u_1(0,t) = (t+\tau)^{\alpha} U,\ u_2(0,t) = (t-\tau)^{\alpha} U,\ u_3(0,t) = e^{\alpha(t+\tau)} U,$$

and

$$u_i(x,t) \to 0 \text{ as } t \to \infty,\ i = 1,2,3,$$

for fixed $t \in [0,T]$.

It was Barenblatt [4], who first discussed the similarity solution u_1; he did this for $\alpha \geq 0$. In a subsequent paper [5] he also investigated the solution u_3 for $\alpha > 0$ and $m = 2$. Later Marshak [14] also discussed solution u_3; in addition he made a detailed, and partly numerical, study of solution u_1 for $\alpha = \frac{1}{5}$. For a number of values of α, explicit solutions were found by various authors [1,4,5,13–15,21].

In this work we discuss the existence and uniqueness of a most general form of solutions (1.2)–(1.4) to equation (1.1), which are written in the form:

$$u\,(x,t) = c\,(t)\,f(\eta),\ \text{with } \eta = \frac{x-b(t)}{a\,(t)},\ a,\ c,\ b \in \mathbb{R}^{+}, \tag{1.8}$$

where $a(t)$, $c(t)$, $b(t)$ and the profile f are to be determined. By replacing this form of solution in this equation, we obtain a general form of nonlinear differential equation which we prove the existence of their solutions under certain conditions. These solutions are called "traveling profiles solutions" [7,8].

2. Traveling profiles solutions to porous medium equation:

If we replace this form of solutions in equation (1.1) we find,

$$\frac{\dot{c}}{c}f - \frac{\dot{a}}{a}\eta f' - \frac{\dot{b}}{a}f' = \frac{c^{m-1}}{a^2}(f^m)'', \tag{2.1}$$

this equation depends on many unknown parameters, our aim is to determine the coefficients $a\,(t)\,, c\,(t)\,, b(t)$ and to prove the existence of the profile f.
In that case, a simple separation of variables argument implies that the following three conditions must hold:

$$\begin{cases} \frac{\dot{c}}{c} &= \frac{c^{m-1}}{a^2}\alpha \\ \frac{\dot{a}}{a} &= -\frac{c^{m-1}}{a^2}\beta \\ \frac{\dot{b}}{a} &= -\frac{c^{m-1}}{a^2}\gamma \end{cases} \tag{2.2}$$

with parameters $\alpha, \beta, \gamma \in \mathbb{R}$, and the profile f must satisfy the equation

$$(f^m)^{''}(\eta) = \alpha f(\eta) + \beta\eta f^{'}(\eta) + \gamma f^{'}(\eta). \tag{2.3}$$

2.1. Resolution of the differential system:

At the boundaries, we impose the lateral boundary conditions

$$a\,(0) = 1, \;\; c\,(0) = 1, \;\; b(0) = 0, \tag{2.4}$$

we can see that from (2.2), we have

$$\begin{cases} c(t) = a(t)^{\frac{-\alpha}{\beta}} \\ b(t) = \frac{\gamma}{\beta}a(t) + K_2 \end{cases}, \tag{2.5}$$

if we replace (2.5) in (2.2), then we deduct

$$\begin{cases} a(t) = (1 - A\beta t)^{\frac{1}{A}} \\ c(t) = (1 - A\beta t)^{\frac{-\alpha}{\beta A}} \\ b(t) = \frac{\gamma}{\beta}\,(1 - A\beta t)^{\frac{1}{A}} - \frac{\gamma}{\beta} \end{cases}, \text{ for } 0 < t < T, \tag{2.6}$$

with

$$2\beta + (m-1)\alpha > 0, \tag{2.7}$$

and

$$A = 2 + \frac{\alpha}{\beta}(m-1), \tag{2.8}$$

the finite time T is given by

$$T = \frac{1}{2\beta + (m-1)\alpha}. \tag{2.9}$$

In other hand, we have

$$\begin{cases} a(t) = \exp\left(-\beta t\right) \\ c(t) = \exp\left(\alpha t\right) \\ b(t) = \frac{\gamma}{\beta}\exp\left(-\beta t\right) - \frac{\gamma}{\beta} \end{cases}, \text{ for } 0 < t < \infty, \tag{2.10}$$

with

$$2\beta + (m-1)\alpha = 0. \tag{2.11}$$

Now we want to prove the existence of the profile f of equation (2.3).

3. Existence and uniqueness of the "based profile":

In this section, we discuss the existence and uniqueness of weak solutions with compact support for the boundary value problem

$$\left(f^m\right)''_{\eta\eta} = \alpha f + \beta\eta f'_\eta + \gamma f'_\eta, \quad 0 < \eta < \infty, \text{ where } \alpha, \beta, \gamma \in \mathbb{R}, \tag{3.1}$$

with $\eta = \frac{x-b(t)}{a(t)}$, and

$$f\left(0\right) = V \quad \text{and} \quad f\left(\infty\right) = 0, \tag{3.2}$$

where $V > 0$ are arbitrary real constants. With this equation for $\gamma = 0$, we recover the forms (1.5)-(1.7) which has been investigated in detail in a series papers (Gilding and Peletier, 1976,1977; Gilding 1980, [12]).
Thus the solution $u(x,t)$ satisfy the lateral boundary condition

$$u(b\left(t\right), t) = c\left(t\right)V, \text{ with } V \in \mathbb{R}^+, \tag{3.3}$$

to the porous medium equation 1.1 in the domain $b(t) < x < \infty, \ t > 0$

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}\left(u^m\right), \quad b(t) < x < \infty, \ t > 0. \tag{3.4}$$

Our aim is to generalize the results of [12] for $\gamma \neq 0$, we follow definition.

Definition 3.1. A function f is a weak solution of (3.1) if it satisfies the following conditions:
a) f is bounded, continuous, and nonnegative on $[0,\infty)$.
b) $\left(f^m\right)(\eta)$ has a continuous derivative with respect to η on $(0,\infty)$.
c) f satisfies the equation

$$\int_0^\infty \phi' \left\{\left(f^m\right)' - \left(\beta\eta + \gamma\right)f\right\} d\xi + (\alpha - \beta)\int_0^\infty \phi f d\eta = 0,$$

for all $\phi \in C_0^1(0,\infty)$.

We will prove the following theorem:

Theorem 3.2. *Suppose that $V > 0$. Then the boundary value problem (3.1)-(3.2) has a weak solution with compact support if and only if $\beta \leq 0,\ \gamma \leq 0$ and $\alpha - 2\beta > 0$. Furthermore, this weak solution is unique.*

To prove this theorem, we impose the following boundary value for equation (3.1):

$$f\left(0\right) = V, \tag{3.5}$$

and

$$f\left(\lambda\right) = 0, \quad \left(f^m\right)^{'}\left(\lambda\right) = 0, \tag{3.6}$$

where $\lambda > 0$ is a real number. Using a shooting argument with $\lambda > 0$ as the shooting parameter, we first prove the following theorem for the existence and uniqueness of classical solutions for (3.1) with the boundary conditions (3.5)-(3.6).

Theorem 3.3. *Suppose that $V > 0$. Then the boundary value problem (3.1), (3.5) and (3.6) has a unique solution and there exists a unique $\lambda(V) > 0$ such that $f(\eta; \lambda(V))$ is positive on $(0, \lambda)$ if and only if $\beta \leq 0,\ \gamma \leq 0$ and $\alpha - 2\beta > 0$.*

We first determine necessary conditions on the parameters $\alpha,\ \beta$ and γ for the existence of a nontrivial weak solution of (3.1) with compact support.

Lemma 3.4. *There exists a nontrivial weak solution of (3.1) with a compact support only when $\beta = \gamma = 0$ and $\alpha > 0$ or $\beta < 0$ and $\gamma < 0$.*

Proof. Suppose that $f(\eta; \lambda)$ is a nontrivial weak solution of (3.1) with compact support. Then $f > 0$ in $(\lambda - \varepsilon, \lambda)$ and $f = 0$ in $[\lambda, \infty)$ for some $\lambda > 0$ and $\varepsilon > 0$.
It follows that f is a classical solution of (3.1) on $(\lambda - \varepsilon, \lambda)$ and satisfies (3.6) at $\eta = \lambda$, that is, $f(\lambda) = 0$, $\left(f^m\right)^{'}(\lambda) = 0$. Integrating (3.1) from η to λ, where $\lambda - \varepsilon < \eta < \lambda$, we get:

$$-\left(f^m\right)^{'}\left(\eta\right) = -\left(\beta\eta + \gamma\right) f\left(\eta\right) + \left(\alpha - \beta\right) \int_{\eta}^{\lambda} f\left(\xi\right) d\xi. \tag{3.7}$$

The continuity of f and $\left(f^m\right)^{'}$ ensures the existence of $\eta_0 \in (\lambda - \varepsilon, \lambda)$ such that $f^{'}\left(\eta_0\right) < 0$. This implies that the LHS of (3.7) is positive at $\eta = \eta_0$, and, therefore, $-\left(\beta\eta_0 + \gamma\right)$ and $\alpha - \beta$ cannot both be less than zero. Thus, $\beta = \gamma = 0$ implies that $\alpha > 0$.

Now consider the case $\beta > 0$ and $\gamma > 0$. This requires that $\alpha - \beta > 0$, and hence $\alpha > 0$. We easily check from (3.1) that f cannot have a maximum as long as f is positive. Therefore, f does not assume a maximum at any point in $(\lambda - \varepsilon, \lambda)$, thus, $f^{'}\left(\eta_0\right) < 0$ on $(\lambda - \varepsilon, \lambda)$. It follows from (3.7) that

$$-mf^{m-2}\left(\eta\right) f^{'}\left(\eta\right) + \left(\beta\eta + \gamma\right)\eta \leq \left(\alpha - \beta\right)\left(\lambda - \eta\right), \tag{3.8}$$

where we have used the fact that $f(\xi) \leq f(\eta)$ for $\xi \in (\eta, \lambda),\ \lambda - \varepsilon < \eta < \lambda$. As $\eta \to \lambda$ in (3.8), LHS becomes positive, and the RHS tends to zero, a contradiction.
Thus we have shown that $\beta = \gamma = 0$ and $\alpha > 0$ or $\beta < 0$ and $\gamma < 0$ are the only cases for which a nontrivial weak solution of (3.1) exists with a compact support. □

3.1. The case when $\beta = \gamma = 0$ and $\alpha > 0$

With $\beta = \gamma = 0$ and $\alpha > 0$, the solution of (3.1), were obtained by Gilding and Peletier, see [12]), and are given by

$$f(\eta;\lambda) = \left[\frac{\alpha\,(m-1)^2}{2m\,(m+1)}\,(\lambda-\eta)^2\right]^{\frac{1}{m-1}},\ 0<\eta<\lambda, \tag{3.9}$$

which is the unique solution of the problem (3.1) satisfying (3.6).
We observe that

$$f(0;\lambda) = \left[\frac{\alpha\,(m-1)^2}{2m\,(m+1)}\lambda^2\right]^{\frac{1}{m-1}}.$$

Because $m > 1$, $f(0;\lambda)$ is a continuous function of λ with $f(0;0) = 0$ and $f(0;\infty) = \infty$; furthermore, f is a continuous and monotonically increasing function of a. This implies that, for a given $V > 0$, there exists a unique $\lambda(V)$ such that $f(0;\lambda(V)) = V$. Therefore, $f(\eta;\lambda(V))$ is the unique solution of (3.1) satisfying (3.5) and (3.6). An easy calculation shows that

$$\lambda(V) = \left[\frac{2m\,(m+1)}{\alpha\,(m-1)^2}V^{m-1}\right]^{\frac{1}{2}}.$$

3.2. The case when $\beta < 0$ and $\gamma < 0$

We give below an elementary lemma for the case $\beta < 0$ and $\gamma < 0$.

Lemma 3.5. *Suppose that $0 < \mu < \lambda$ and f is a positive solution of (3.1) on $[\mu,\lambda)$ satisfying (3.6). Then the following results hold.:*
(i) *$f^{'}(\eta) < 0$ on $[\mu,\lambda)$ provided that $\alpha - \beta \geq 0$.*
(ii) *Suppose that $\alpha - \beta < 0$ and $f^{'}(\eta_0) = 0$ for some $\eta_0 \in [\mu,\lambda)$. Then f has a maximum at η_0 for $\eta_0 < \frac{\lambda(\alpha-\beta)-\gamma}{\alpha}$.*
Suppose that f is a positive solution of (3.1) and (3.6) on $[0,\lambda)$. Then

$$f^{'}(0) < 0,\ \textit{for}\ \alpha-\beta \geq 0.$$

Proof.
(i) By integration of (3.1) from $\mu < \eta < \lambda$, we obtain

$$-\left(f^m\right)^{'}(\eta) = -\left(\beta\eta+\gamma\right)f\left(\eta\right) + \left(\alpha-\beta\right)\int_\eta^\lambda f\left(\xi\right)d\xi. \tag{3.10}$$

Because $\beta < 0$ and $\gamma < 0$, the RHS of (3.10) is positive when $\alpha - \beta \geq 0$ and hence $\left(f^m\right)^{'}(\eta) < 0$. This implies that $f^{'}(\eta) < 0$ on $[\mu,\lambda)$.
(ii) if $\alpha - \beta < 0$ then $\alpha < 0$ (because $\beta < 0$), by (3.1), $f^{''}(\eta_0) < 0$ when $f^{'}(\eta_0) = 0$,

thus f has a maximum at $\eta = \eta_0$ and is strictly decreasing on (η_0, λ); that is, $f^{'}(\eta) < 0$ on (η_0, λ). Putting $\eta = \eta_0$ in (3.10), we have:

$$0 = -(\beta\eta_0 + \gamma) f(\eta_0) + (\alpha - \beta) \int_{\eta_0}^{\lambda} f(\xi)\, d\xi > -(\beta\eta_0 + \gamma) f(\eta_0) + (\alpha - \beta)(\lambda - \eta_0) f(\eta_0),$$

therefore,

$$-(\beta\eta_0 + \gamma) + (\alpha - \beta)(\lambda - \eta_0) < 0 \text{ or } \eta_0 < \frac{\lambda(\alpha - \beta) - \gamma}{\alpha},$$

With $\eta = 0$, (3.10) becomes

$$-(f^m)^{'}(0) = -\gamma f(0) + (\alpha - \beta) \int_{\eta}^{\lambda} f(\xi)\, d\xi. \tag{3.11}$$

The result for $f^{'}(0)$ follows immediately from (3.11). □

In the next lemma, we prove the local existence and uniqueness of a solution of (3.1) satisfying (3.6). This is accomplished by formulating an equivalent integral equation following the work of Atkinson and Peletier [3].

Lemma 3.6. *Suppose that $\beta < 0$, $\gamma < 0$ and α is any real number. Then, for any $\lambda > 0$, equation (3.1) with initial condition (3.6) at $\eta = \lambda$, has a unique positive solution in a neighborhood $(\lambda - \varepsilon, \lambda)$ of λ, here, $\varepsilon > 0$ is a constant.*

Proof. Suppose that f is a positive solution in a left neighborhood of $\eta = \lambda$. By lemma 3.5, $f^{'}(\eta) < 0$ for $\eta \in (\lambda - \varepsilon, \lambda)$ for some $\varepsilon > 0$.
Let $\eta = G(f)$ where G is the inverse of f on $(\lambda - \varepsilon, \lambda)$. Rewriting (3.10), we have:

$$(f^m)^{'}(\eta) = (\alpha\eta + \gamma) f(\eta) + (\alpha - \beta) \int_{\eta}^{\lambda} \xi f^{'}(\xi)\, d\xi. \tag{3.12}$$

With $G(f) = \eta$ in (3.12) we have:

$$\frac{dG}{df} = \frac{m f^{m-1}}{(\alpha G + \gamma) f - (\alpha - \beta) \int_0^f G(\varphi) d\varphi}, \tag{3.13}$$

equation (3.13) is an integro-differential equation for $G = G(f)$. Integrating (3.13) from zero to f , we obtain

$$G(f) - \lambda = m \int_0^f \frac{\phi^{m-1} d\phi}{(\alpha G + \gamma) \phi - (\alpha - \beta) \int_0^{\phi} G(\psi) d\psi}. \tag{3.14}$$

Let

$$H(f) = 1 - \lambda^{-1} G(f), \tag{3.15}$$

Then, equation (3.14) becomes

$$H(f)=\frac{m}{\lambda^2}\int_0^f \frac{\phi^{m-1}d\phi}{(-\beta-\gamma)\,\phi+\alpha\phi H(\phi)-(\alpha-\beta)\int_0^\phi H(\psi)d\psi}. \tag{3.16}$$

By using the Banach–Cacciopoli contraction mapping principle, we now show that equation (3.16) admits a unique positive solution in a right neighborhood of $f=0$. Let X be the set of all bounded functions $H(f)$ on $[0,h], h>0$, satisfying

$$0\leq H(f)\leq \rho=\frac{|\beta+\gamma|}{2(|\alpha|+|\alpha-\beta|)}. \tag{3.17}$$

Let $\|..\|$ be the sup norm defined on X. Then X is a complete metric space.

$$M\,(H)\,(f)=\frac{m}{\lambda^2}\int_0^f \frac{\phi^{m-1}d\phi}{-\,(\beta+\gamma)\,\phi+\alpha\phi H(\phi)-(\alpha-\beta)\int_0^\phi H(\psi)d\psi},\ \ H(f)\in X. \tag{3.18}$$

First we show that M maps X into X over $[0,h_0]$, $h\leq h_0$. Let $H\in X$. Clearly,

$$\begin{aligned}-\,(\beta+\gamma)\,\phi+\alpha\phi H(\phi)-(\alpha-\beta)\int_0^\phi H(\psi) &\geq -\,(\beta+\gamma)\,\phi-|\alpha|\,\phi H(\phi) &(3.19)\\ &- |\alpha-\beta|\,\|H\|\,\phi\\ &\geq -\,(\beta+\gamma)\,\phi &(3.20)\\ &- (|\alpha|+|\alpha-\beta|)\,\|H\|\,\phi\\ &\geq \frac{-\,(\beta+\gamma)\,\phi}{2}, &(3.21)\end{aligned}$$

where we have used (3.17). Therefore, from (3.18), we have

$$\begin{aligned}M\,(H)\,(f) &\leq \frac{2m}{-\,(\beta+\gamma)\,\lambda^2}\int_0^f \phi^{m-2}d\phi\\ &= \frac{2mf^{m-2}}{-\,(\beta+\gamma)\,\lambda^2\,(m-1)}\\ &\leq \frac{2mh^{m-2}}{-\,(\beta+\gamma)\,\lambda^2\,(m-1)}. &(3.22)\end{aligned}$$

Thus, $M(H)$ is well defined on X and $M(H):[0,h]\to\mathbb{R}$ is nonnegative and continuous. The RHS of (3.22) suggests that we may find $h_0,\ h\leq h_0$ such that $\|M(H)\|\leq\rho,\ H\in X$. Thus M maps X into X for $h\leq h_0$. In the next step, we show that M is a contraction map on X. Let $H_1,H_2\in X$, and $h\leq h_0$.

Then

$$\begin{aligned}\|M(H_1)-M(H_2)\| &\leq \frac{4m}{(\beta+\gamma)^2\lambda^2}\int_0^f \phi^{m-3}\Big(|\alpha|\,\phi\,\|H_1-H_2\| \\ &+ |\alpha-\beta|\int_0^\phi \|H_1-H_2\|\,d\psi\Big)d\phi \\ &\leq \frac{4m}{(m-1)(\beta+\gamma)^2\lambda^2}\left(|\alpha|+|\alpha-\beta|\right)h^{m-1}\,\|H_1-H_2\|\,.\end{aligned}$$

Therefore, there exists $h_1\in(0,h_0]$ such that if $h\leq h_1$, M is a contraction on X. By the Banach–Cacciopoli contraction principle, M has a unique fixed point in X and hence equation (3.16) has a unique solution. This, in turn, implies that there exists a unique positive solution of (3.1)-(3.6) in an interval $(\lambda-\varepsilon,\lambda)$ for some $\varepsilon>0$. □

In the next lemma, we prove that a positive solution $f(\eta;\lambda)$ of (3.1) and (3.6) cannot be unbounded.

Lemma 3.7. *Suppose that $\beta<0$, $\gamma<0$ and $\mu\in[0,\lambda)$. Furthermore, let f be a positive solution of (3.1) and (3.6) on (μ,λ). Then f is bounded on (μ,λ) and*

$$\sup f(\eta)\leq\left[\frac{(m-1)\lambda}{2m}\max\left\{-(\beta\lambda+2\gamma),\,[(\alpha-2\beta)\lambda-2\gamma]\right\}\right]^{\frac{1}{m-1}}.$$

Proof. We prove this lemma for the two following cases: (i) $\alpha-\beta\geq0$, (ii) $\alpha-\beta<0$.
Case $\alpha-\beta\geq0:$
In this case, $f^{'}(\eta)<0$ on (μ,λ) by Lemma 3.5, $f(\eta)\geq f(\xi)\,,\ \xi\in(\eta,\lambda)$. By (3.10), we have

$$-(f^m)^{'}(\eta)\leq-(\beta\eta+\gamma)f(\eta)+(\alpha-\beta)f(\eta)(\lambda-\eta)\,,\ \mu\leq\eta<\lambda,$$

or

$$-mf^{m-2}f^{'}\leq-\alpha\eta-\gamma+\lambda(\alpha-\beta)\leq-\lambda\beta-\gamma+\alpha(\lambda-\eta)\,,\ \mu\leq\eta<\lambda. \tag{3.23}$$

Integrating (3.23) from η to λ, we obtain

$$\frac{m}{m-1}f^{m-1}(\eta)\leq\left[-\lambda\beta-\gamma+\frac{1}{2}\alpha(\lambda-\eta)\right](\lambda-\eta)\,,\ \mu\leq\eta\leq\lambda. \tag{3.24}$$

Thus

$$\frac{m}{m-1}\sup_{(\mu,\lambda)}f^{m-1}(\eta)\leq\frac{1}{2}\left[(\alpha-2\beta)\lambda-2\gamma\right]\lambda. \tag{3.25}$$

Case $\alpha-\beta<0:$
By equation (3.10),

$$-(f^m)^{'}(\eta)\leq-(\beta\eta+\gamma)f(\eta)\,,\ \mu\leq\eta<\lambda,$$

or

$$-mf^{m-2}f^{'} \leq -(\beta\eta+\gamma),\ \mu \leq \eta < \lambda. \tag{3.26}$$

Integrating (3.26) from η to λ, we obtain

$$\frac{m}{m-1}f^{m-1}(\eta) \leq -\left[\frac{\beta}{2}\left(\lambda^2-\eta^2\right)+\gamma(\lambda-\eta)\right],\ \mu \leq \eta \leq \lambda. \tag{3.27}$$

This in turn, implies that

$$\frac{m}{m-1}\sup_{(\mu,\lambda)} f^{m-1}(\eta) \leq -\frac{\lambda}{2}(\beta\lambda+2\gamma). \tag{3.28}$$

Observe that the bounds in (3.25) and (3.28) are independent of μ and, therefore, $f(\eta)$ cannot be unbounded as η decreases from $\eta=\lambda$. □

Lemma 3.8. *Suppose that f is a positive solution of (3.1) and (3.6) in a left neighborhood of $\eta=\lambda$, and $\beta<0,\ \gamma<0$. Then $f(\eta)>0$ on $[0,\lambda)$ when $\alpha-2\beta>0$.*

Proof. Integrating (3.10) from η to λ we have

$$f^m(\eta) = -(\beta\eta+\gamma)\int_\eta^\lambda f(\xi)\,d\xi + (\alpha-2\beta)\int_\eta^\lambda (\xi-\eta)f(\xi)\,d\xi. \tag{3.29}$$

It is easy to see from (3.29) that, if $\alpha-2\beta>0$, then $f(\eta)>0$ on $(0,\lambda)$. □

Prove of Theorem 3.3. Now we proceed to prove Theorem 3.3. We have already proved in Lemma 3.6 the local existence of a solution about $\eta=\lambda$ for (3.1) and (3.6). This unique local solution may be extended back to $\eta=0$ as a positive solution with $f(0)>0$ if and only if when $\alpha-2\beta>0$ (see Lemma 3.8). Now if we can prove that there exists $\lambda(V)$ such that $f(0;\lambda(V))=V$, then Theorem 3.3 is proved. To that end, we use the following result due to Barenblatt (see [5]). Suppose that $f(\eta;\lambda)$ is a solution of (3.1) and (3.6) on $(0,\lambda)$; then $\omega^{-\frac{2}{m-1}}f(\omega\eta;\omega\lambda)$ is a solution of (3.1) and (3.6) on $(0,\omega\lambda)$ for any $\omega>0$. Let $\omega=\lambda^{-1}$, then,

$$f(0;\lambda)=\lambda^{\frac{2}{m-1}}f(0;1)=V. \tag{3.30}$$

Because $f(0;1)>0$ for $\alpha-2\beta>0,\ \beta<0,\ \gamma<0$, we get a unique root $\lambda=\lambda(V)$ of (3.30). Thus, $f(\eta;\lambda(V))$ is the unique solution of (3.1), (3.5) and (3.6).

Theorem 3.3 follows if we add that, for $\beta=\gamma=0$, we have already constructed the explicit solution (3.10):

$$f(\eta;\lambda)=\left[\frac{\alpha(m-1)^2}{2m(m+1)}(\lambda-\eta)^2\right]^{\frac{1}{m-1}},\ 0<\eta<\lambda.$$

□

Prove of Theorem 3.2. We observe that

$$f(\eta) = \begin{cases} f(\eta;\lambda), & 0 < \eta < \lambda \\ 0, & \lambda < \eta < \infty \end{cases}, \tag{3.31}$$

is a weak solution of (3.1) and (3.6). Now we must show that, given $V > 0$, (3.31) is the only solution of (3.1), (3.5) and (3.6) with compact support.
Suppose that $f(\eta)$ is a weak solution of the problem (3.1) and (3.2) with compact support. By Lemma 3.8, this is possible only if $\alpha - 2\beta > 0$. Moreover,

$$f(\eta) \begin{cases} > 0, & \text{on} \quad \eta \in [0, \lambda)\,, \\ = 0, & \text{on} \quad \eta \in [\lambda, \infty)\,, \ \lambda > 0. \end{cases}$$

By Theorem 3.3, this is also the unique solution. Thus, we have proved Theorem 3.2. □

We conclude with a discussion of the implications of Theorems 3.2 and 3.3 for general form of self similar solutions to equation (1.1).

Theorem 3.9. *If $\beta < 0,\ \gamma < 0$ and $\alpha \geq \frac{2\beta}{1-m}$,*
the problem (3.4), (3.3) has a weak solution with compact support in the form

$$u\,(x,t) = c\,(t)\, f\,(\eta)\,, \ \textit{with}\ \eta = \frac{x - b(t)}{a\,(t)},$$

where the "based profile" f is a solution of following differential equation

$$(f^m)''_{\eta\eta} = \alpha f + \beta\eta f'_\eta + \gamma f'_\eta, \quad 0 < \eta < \infty.$$

and the coefficients $c\,(t)\,, a\,(t)$ and $b(t)$ are given by
1)

$$\begin{cases} a(t) = (1 - A\beta t)^{\frac{1}{A}} \\ c(t) = (1 - A\beta t)^{\frac{-\alpha}{\beta A}} \\ b(t) = \frac{\gamma}{\beta}\,(1 - A\beta t)^{\frac{1}{A}} - \frac{\gamma}{\beta} \end{cases}, \ 0 < t < T,$$

if $\alpha > \frac{2\beta}{1-m}$, $A = 2 + \frac{\alpha}{\beta}(m-1)$ with

$$T = \frac{1}{2\beta + (m-1)\alpha},$$

and by
2)

$$\begin{cases} a(t) = \exp\,(-\beta t) \\ c(t) = \exp\,(\alpha t) \\ b(t) = \frac{\gamma}{\beta} \exp\,(-\beta t) - \frac{\gamma}{\beta} \end{cases}, \ 0 < t < \infty.$$

if $\alpha = \frac{2\beta}{1-m}$.

Proof. We have already proved in Theorem 3.3 the existence of "based profile" f with compact support if and only if $\beta < 0,\ \gamma < 0$ and $\alpha - 2\beta > 0$.
The coefficients $c\,(t)\,, a\,(t)$ and $b(t)$ are given by (2.6)

$$\begin{cases} a(t) = (1 - A\beta t)^{\frac{1}{A}} \\ c(t) = (1 - A\beta t)^{\frac{-\alpha}{\beta A}} \\ b(t) = \frac{\gamma}{\beta}\,(1 - A\beta t)^{\frac{1}{A}} - \frac{\gamma}{\beta} \end{cases} \quad , \text{ for } 0 < t < T$$

with $A = 2 + \frac{\alpha}{\beta}(m-1)$ and $T = \frac{1}{2\beta+(m-1)\alpha} > 0$, then $2\beta + (m-1)\alpha > 0$, ie $\alpha > \frac{2\beta}{1-m}$.
Clearly the coefficients $c(t), a(t)$ and $b(t)$ are defined if $1 - A\beta t > 0$, this implies

$$t < \frac{1}{A\beta} = \frac{1}{2\beta + (m-1)\alpha} = T.$$

We see that the solution $u\,(x,t)$ blows up at $t = T$. and $T = \frac{1}{2\beta+(m-1)\alpha}$, is the blow-up time, such that the solution is well defined for all $0 < t < T$, while $u\,(x,t) \to \infty$ as $t = T$.
We recover also (2.10)

$$\begin{cases} a(t) = \exp\left(-\beta t\right) \\ c(t) = \exp\left(\alpha t\right) \\ b(t) = \frac{\gamma}{\beta}\exp\left(-\beta t\right) - \frac{\gamma}{\beta} \end{cases} \quad , \text{ for } 0 < t < \infty.$$

if $2\beta + (m-1)\alpha = 0$, ie $\alpha = \frac{2\beta}{1-m}$.
Finally, we have proved that solution $u\,(x,t) = c\,(t)\,f\left[\frac{x-b(t)}{a(t)}\right]$ exists for $\beta < 0,\ \gamma < 0$ and $\alpha \geq \frac{2\beta}{1-m}$. □

Conclusion

In this work we have proved the existence of some class of solutions called "traveling profiles solutions" to the porous medium equation in one dimension. We have generalized the results obtained by Gilding who proved the existence of weak solutions with compact support under self similar form. we have also found new exact solutions of porous media equation in our general form of self similar solutions.

Acknowledgment. The authors are deeply grateful to the reviewers and editors for their insightful comments that have helped to improve the quality of this research work that is supported by the General Direction of Scientic Research and Technological Development (DGRSTD) – Algeria.

References

[1] W.F. Ames, *Similarity for the nonlinear diffusion equation*, I&f EC Fundamentals 4 (1965) 72–16.

[2] D.G. Aronson, L.A. Peletier, *Large time behaviour of solutions of the porous medium equation in bounded domains*, J. Diff. Eq. 39 (1981) 378–412.

[3] F.V. Atkinson, L.A. Peletier, *Similarity profiles of flows through porous media*, Arch. Rational Mech. Anal. 42 (1971) 369–379.

[4] G.I. Barenblatt, *On some unsteady motions of a liquid or a gas in a porous medium*, Prikl. Mat. Meh. 16 (1952) 67–78.

[5] G.I. Barenblatt, *On limiting self-similar motions in the theory of unsteady filtration of a gas in a porous medium and the theory of the boundary layer*, Prikl. Mat. Mek. 18 (1954) 409–414.

[6] G.I. Barenblatt, *Similarity, Self-Similarity and Intermediate Asymptotics*, Consultants Bureau, New York, 1979.

[7] N. Benhamidouche, *Exact solutions to some nonlinear PDEs, travelling profiles method*, Electronic Journal of Qualitative Theory of Differential Equation 15 (2008) 1–7.

[8] Y. Arioua, N. Benhamidouche, *New method for constructing exact solutions to nonlinear PDEs*, International Journal of Nonlinear Science 7 (4) (2009) 395–398.

[9] R. Djemiat, B. Basti, N. Benhamidouche, *Analytical studies on the global existence and blow-up of solutions for a free boundary problem of two dimensional diffusion equations of moving fractional order*, Advances in the Theory of Nonlinear Analysis and its Applications 6 (3) (2022) 287–299.

[10] J.I. Diaz, R. Kersner, *On a nonlinear degenerate parabolic equation in infiltration or evaporation through a porous medium*, J. Differential Equations 69 (1987) 368–403.

[11] R. Chouder, N. Benhamidouche, *Travelling profile solutions for nonlinear degenerate parabolic equation and contour enhancement in image processing*, Applied Mathematics E-Notes 18 (2018) 1–12.

[12] B.E. Gilding, L.A. Peletier, *On a class of similarity solutions of the porous medium equation I, II; and III*, J. Math. Anal. Appl. 55 (1976) 351–364:II, 57 (1977) 522–538, 77, 38 1402 (1980).

[13] L. Matyas, I.F. Barna, *General self-similar solutions of diffusion equation and related constructions*, Romanian Journal of Physics 67 (2022) Article no. 101.

[14] R.E. Marshak, *Effect of radiation on shock wave behaviour*, Pkys. Fluids 1 (1958) 24–29.

[15] O.A. Oleinik, A.S. Kalashnikov, C. Yui-Ian, *The Cauchy problem and boundary problems for equations of the type of unsteady filtration, IXJ*, Akad. Nauk. SSSR Ser. Mat. 22 (1958) 667–704.

[16] L. Paude, J. Iaia, *Traveling wave solutions of the porous medium equation with degenerate interfaces*, Nonlinear Analysis: Theory, Methods & Applications 81 (2013) 110–129.

[17] A.D. Polyanin, A.I. Zhurov, A.V. Vyazmin, *Generalized separation of variables and mass transfer equations*, J. Non Equilib. Thermodyn 25 (2000) 251–267.

[18] P.L. Sachdev, Ch. Srinivasa Rao, *Large Time Asymptotics for Solutions of Nonlinear Partial Differential Equations*, Springer Science+Business Media, LLC, 2010.

[19] J.L. Vazquez, *The Porous Medium Equation: Mathematical Theory*, Clarendon Press, Oxford, 2007.

[20] C.J Van Duyn, J.M De Graaf, *Limiting profiles in contaminant transport through porous media*, SIAM J. Math. Anal 18 (1987) 728–743.

[21] Ya.B. Zeldovich, Yu.P. Raizer, *Physics of Shock-waves and High-temperature Hydrodynamic Phenomena Vol. II*, Academic Press, New York, 1966.

DOI: 10.7862/rf.2023.6

Yacine Arioua
email: yacine.arioua@univ-msila.dz
ORCID: 0000-0002-9681-9568
Faculty of Mathematics and Computer science
Laboratory for Pure and Applied Mathematics
University of M'sila
Bp 166 M'sila, 28000
ALGERIA

Nouredine Benhamidouche
email: nouredine.benhamidouche@univ-msila.dz
ORCID: 0000-0002-5740-8504
Faculty of Mathematics and Computer science
Laboratory for Pure and Applied Mathematics
University of M'sila
Bp 166 M'sila, 28000
ALGERIA

Received 12.01.2022 *Accepted 24.10.2023*