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The Joint Laplace-Hankel Transforms for Fractional Diffusion Equation

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Abstract: Operational methods are used to accomplish the solution of certain problems with less effort and in a simple routine way. Laplace transforms can be used to solve certain types of fractional singular integral equation not considered in the literature. In this study, the author implemented an analytical technique the joint Laplace-Hankel transforms to provide the exact solution for a time fractional non-homogeneous diffusion equation with non-constant coefficients in cylindrical coordinates. The obtained results reveal that the joint transform method is very convenient and effective. Certain non trivial integral identities involving Airy functions and modified Bessel functions of the second kind are also provided.

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1. Introduction and Preliminaries

In recent years, a growing number of research works done by many researchers from various fields of engineering and science deal with dynamical systems described by equations of fractional order which means equations involving derivatives and integrals of fractional order.
In this work, the author studied analytically distribution functions during ion cyclotron resonance heating (ICRH) by using the one-dimensional Fokker-Planck equation incorporating ion-electron and ion-ion collisions and quasi-linear diffusion. In the equation, we include source and loss terms and we find the steady-state and time-dependent solutions which are regular in the origin and vanish at high energies. The

main purpose of the current study is to develop a method for evaluation of certain integrals and finding analytic solutions of fractional PDEs. An analytical technique approaches, the joint Laplace-Hankel transforms to provide the exact solution for a time fractional non-homogeneous diffusion equation with non-constant coefficients in cylindrical coordinates.

1.1. Definitions and Notations

Definition 1.1. With $\mathcal{D}_t^{c,\alpha}$ we denote the time fractional derivative of order α $(0 < \alpha < 1)$ regularized in the Dzhrbashyan-Caputo sense defined for a sufficiently regular function $\phi(t)$, as

$$D_t^{c,\alpha}\phi(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{1}{(t-\xi)^\alpha}\phi'(\xi)d\xi. \tag{1.1}$$

Remark. In this work, we prefer Caputo fractional derivative to Riemann-Liouville one since the former is more popular in real applications. When we adopt the Caputo fractional derivative of order-α, the initial values of $y(0), y'(0), ..., y^m(0)$, where $m = [\alpha]$, are enough. Obviously, these initial values are prone to measure since they have all physical meaning. On the other hand, we choose Caputo fractional derivative due to another fact that the non-homogeneous initial conditions are permitted if such conditions are necessary.

Definition 1.2. The Laplace transform of the function $f(t)$ is given by[1-3]

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt := F(s). \tag{1.2}$$

If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}^{-1}\{F(s)\}$ is as follows

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds, \tag{1.3}$$

where $F(s)$ is analytic in the region $\mathrm{R}e(s) > c$.
The expression in equation (1.3) is the inverse Laplace transform for the function $F(s)$, and is often called the Bromwich integral.

Lemma 1.1. *Let $L\{f(t)\} = F(s)$ then, the following identities hold*

1. $\mathcal{L}^{-1}(\frac{1}{\sqrt{s}(\sqrt{s}+\lambda)}) = e^{\lambda^2 t}Erfc(\lambda\sqrt{t})$,
2. $e^{-\omega s^\beta} = \frac{1}{\pi}\int_0^\infty e^{-r^\beta(\omega\cos\beta\pi)}\frac{\sin(\omega r^\beta \sin\beta\pi)}{s+r}dr$,
3. $\mathcal{L}^{-1}(F(s^\alpha)) = \frac{1}{\pi}\int_0^\infty f(u)\int_0^\infty e^{-tr-ur^\alpha\cos\alpha\pi}\sin(ur^\alpha \sin\alpha\pi)drdu$,
4. $\mathcal{L}^{-1}(F(\sqrt[3]{s})) = \frac{1}{3\pi}\int_0^\infty (\frac{u}{t})^{\frac{3}{2}}K_{\frac{1}{3}}(\frac{2u\sqrt{u}}{3\sqrt{3t}})f(u)du$.

Proof. See [1].

Example 1.1. The fractional integral of order α of the function $\phi(t)$, with $0 < \alpha < 1$ is defined as follows

$$\mathcal{J}^{\alpha}\phi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1}\phi(\xi)d\xi,$$

then the Laplace transform of the fractional integral of order α is as below

$$\mathcal{L}[\mathcal{J}^{\alpha}\phi(t)] = \int_0^{+\infty} e^{-st}[\frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1}\phi(\xi)d\xi]dt = \frac{\Phi(s)}{s^{\alpha}}.$$

Lemma 1.2. *The following integral relation holds*

$$\mathcal{L}^{-1}[\frac{e^{-k\sqrt{s}}}{s^{\nu}+\lambda}; s \to t] = f(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-t\xi}[\frac{\xi^{\nu}\sin(\pi\nu - k\sqrt{\xi}) - \lambda\sin(k\sqrt{\xi})}{\xi^{2\nu}+2\lambda\xi^{\nu}\cos(\pi\nu)+\lambda^2}]d\xi.$$

Proof. In view of the Titchmarch theorem or Gross-Levi lemma [3], we have the following

$$f(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-t\xi} Im[\frac{e^{-k\sqrt{\xi e^{-i\pi}}}}{(\xi e^{-i\pi})^{\nu}+\lambda}]d\xi,$$

or

$$f(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-t\xi} Im[\frac{e^{-ik\sqrt{\xi}}}{\xi^{\nu}(\cos(\pi\nu) - i\sin(\pi\nu))+\lambda}]d\xi,$$

after simplifying we have

$$f(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-t\xi} Im\frac{[\cos(k\sqrt{\xi}) - i\sin(k\sqrt{\xi})][\xi^{\nu}\cos(\pi\nu)+\lambda+i\xi^{\nu}\sin(\pi\nu)]}{\xi^{2\nu}+2\lambda\xi^{\nu}\cos(\pi\nu)+\lambda^2}d\xi,$$

or

$$f(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-t\xi}[\frac{\xi^{\nu}\sin(\pi\nu - k\sqrt{\xi}) - \lambda\sin(k\sqrt{\xi})}{\xi^{2\nu}+2\lambda\xi^{\nu}\cos(\pi\nu)+\lambda^2}]d\xi.$$

Let us consider the special cases
1. $\lambda = k = 0,\ 0 < \nu < 1$ we have

$$\mathcal{L}^{-1}[\frac{1}{s^{\nu}}; s \to t] = f(t) = \frac{\sin(\pi\nu)}{\pi}\int_0^{+\infty} e^{-t\xi}\xi^{-\nu}d\xi = \frac{t^{\nu-1}}{\Gamma(\nu)}.$$

2. $k = 0$, we have

$$\mathcal{L}^{-1}[\frac{1}{s^{\nu}+\lambda}; s \to t] = f(t) = \frac{\sin(\pi\nu)}{\pi}\int_0^{+\infty} [\frac{\xi^{\nu}e^{-t\xi}}{\xi^{2\nu}+2\lambda\xi^{\nu}\cos(\pi\nu)+\lambda^2}]d\xi.$$

$\square$

Lemma 1.3. *The following integral relation holds*

$$\mathcal{L}^{-1}[\frac{1}{s^{\beta}(s^{\nu}+\lambda)};s\to t]=\int_0^t \frac{(t-\eta)^{\beta-1}}{\Gamma(\beta)}\left[\frac{\sin(\pi\nu)}{\pi}\int_0^{+\infty}[\frac{\xi^{\nu}e^{-\eta r}}{\xi^{2\nu}+2\lambda r^{\nu}\cos(\pi\nu)+\lambda^2}]d\xi\right]d\eta.$$

Proof. Making use of the convolution theorem for the Laplace transform.

Corollary 1.1. *Let us show that*

$$\mathcal{L}^{-1}[\frac{\pi}{\sqrt{3}}e^{-3\sqrt[3]{s}};s\to t]=t^{-\frac{3}{2}}K_{\frac{1}{3}}(\frac{2}{\sqrt{t}}).$$

Note. In the above relation $K_{\nu}(.)$ stands for the modified Bessel function of the second kind of order ν.

Proof. Let us choose $f(u)=\delta(u-\lambda)$ then we have $F(s)=e^{-\lambda s}$, in view of part four of the Lemma 1.1. we get

$$\mathcal{L}^{-1}[e^{-3\sqrt[3]{s}};s\to t]=\frac{1}{3\pi}\int_0^{\infty}(\frac{u}{t})^{\frac{3}{2}}K_{\frac{1}{3}}(\frac{2u\sqrt{u}}{3\sqrt{3t}})\delta(u-\lambda)du=\frac{1}{3\pi}(\frac{\lambda}{t})^{\frac{3}{2}}K_{\frac{1}{3}}(\frac{2\lambda\sqrt{\lambda}}{3\sqrt{3t}}).$$

If we choose $\lambda=3$, after simplifying we arrive at

$$\mathcal{L}^{-1}[\frac{\pi}{\sqrt{3}}e^{-3\sqrt[3]{s}};s\to t]=t^{-\frac{3}{2}}K_{\frac{1}{3}}\left(\frac{2}{\sqrt{t}}\right).$$

In the above relation if we set $s=0$ we have

$$\int_0^{+\infty}t^{-\frac{3}{2}}K_{\frac{1}{3}}(\frac{2}{\sqrt{t}})dt=\int_0^{+\infty}K_{\frac{1}{3}}(\xi)d\xi=\frac{\pi}{\sqrt{3}}.$$

□

Theorem 1.1. *Let us consider fractional singular integro-differential equation*

$$D_{0,t}^{c,\alpha}\phi(t)=f(t)+\lambda\int_t^{+\infty}\phi(\xi)d\xi,\quad 0<t<+\infty$$

$$\phi(0)=u_0,\quad \int_0^{+\infty}\phi(\xi)d\xi=k,\quad 0<\alpha<1,$$

then, the above fractional singular integro-differential equation has the following formal solution

$$\phi(t)=u_0\sum_{n=0}^{+\infty}\frac{(-\lambda)^n t^{(\alpha+1)n}}{\Gamma(1+(\alpha+1)n)}+\sum_{n=0}^{+\infty}(-\lambda)^n\int_0^t f(t-\eta)\frac{\eta^{(\alpha+1)n}}{\Gamma(1+(\alpha+1)n)}d\eta$$

$$-\lambda k\sum_{n=0}^{+\infty}\frac{(-\lambda)^n t^{(\alpha+1)(1+n)-1}}{\Gamma(1+(\alpha+1)n)}.$$

Note. To the best of the author's knowledge this kind of singular integral equation is not considered in the literature.
Solution. Taking the Laplace transform of the above fractional singular integral equation term wise, leads to

$$s^{\alpha}\Phi(s) - s^{\alpha-1}u_0 = F(s) + \lambda\frac{\Phi(s) - \Phi(0)}{s} = F(s) + \lambda\frac{\Phi(s) - k}{s}.$$

After solving the above equation, we obtain

$$\Phi(s) = \frac{sF(s)}{\lambda + s^{\alpha+1}} + \frac{u_0 s^{\alpha} - \lambda k}{\lambda + s^{\alpha+1}},$$

or

$$\Phi(s) = \sum_{n=0}^{+\infty}(-\lambda)^n\left[\frac{F(s)}{s^{n(\alpha+1)+\alpha}} + \frac{u_0}{s^{(\alpha+1)n+1}} - \frac{\lambda k}{s^{(\alpha+1)(n+1)}}\right].$$

At this point, taking the inverse Laplace transform term-wise, we arrive at

$$\phi(t) = \sum_{n=0}^{+\infty}(-\lambda)^n[\int_0^t f(t-\xi)\frac{\xi^{n(\alpha+1)+\alpha-1}}{\Gamma(n(\alpha+1)+\alpha)}d\xi$$

$$+\frac{u_0 t^{(\alpha+1)n}}{\Gamma(n(\alpha+1)+1)} - \frac{\lambda k t^{(\alpha+1)(n+1)-1}}{\Gamma((\alpha+1)(n+1))}], \quad 0 < t < +\infty.$$

It is easy to verify that $\phi(0) = u_0$.
Let us consider the special case $\alpha = 0.5$, we have

$$\phi(t) = \sum_{n=0}^{+\infty}(-\lambda)^n[\int_0^t f(t-\xi)\frac{\xi^{\frac{3n-1}{2}}}{\Gamma(\frac{3n+1}{2})}d\xi$$

$$+\frac{u_0 t^{\frac{3n}{2}}}{\Gamma(\frac{3n}{2}+1)} - \frac{\lambda k t^{\frac{3n+1}{2}}}{\Gamma((\frac{3}{2}(n+1))}], \quad 0 < t < +\infty.$$

Example 1.2. Let us assume that

$$\Psi_n(s) = \int_0^{+\infty}(\xi^2+1)^{\frac{n-1}{2}}e^{-s\sqrt{\xi^2+1}}d\xi$$

then we have

$$\mathcal{L}^{-1}[\Psi_n(s); s \to t] = \frac{t^{\frac{n+1}{2}}}{\sqrt{t^2-1}}.$$

Proof. Let us start with the integral representation of $\Psi_1(s)$, $\Psi_1(s)=\int_0^{+\infty} e^{-s\sqrt{\xi^2+1}}d\xi$, then taking n-times derivative with respect to parameter s leads to

$$\Psi_n(s) = \int_0^{+\infty} (\xi^2+1)^{\frac{n-1}{2}} e^{-s\sqrt{\xi^2+1}} d\xi.$$

By taking inverse Laplace transform followed by the complex inversion formula, we have

$$\psi_n(t) = \mathcal{L}^{-1}[\Psi_n(s); s \to t] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}[\int_0^{+\infty} (\xi^2+1)^{\frac{n-1}{2}} e^{-s\sqrt{\xi^2+1}} d\xi]ds.$$

At this stage changing the order of integration leads to

$$\phi_n(t) = \mathcal{L}^{-1}[\Psi_n(s); s \to t] = \int_0^{+\infty} (\xi^2+1)^{\frac{n-1}{2}} [\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{(t-\sqrt{\xi^2+1})s} ds]d\xi.$$

The value of the inner integral is $\delta(t-\sqrt{\xi^2+1})$, we arrive at

$$\psi_n(t) = \mathcal{L}^{-1}[\Psi_n(s); s \to t] = \int_0^{+\infty} (\xi^2+1)^{\frac{n-1}{2}} \delta(t-\sqrt{\xi^2+1}) d\xi.$$

In order to evaluate the above integral, we make a change of variable

$$t-\sqrt{\xi^2+1} = \eta$$

$$\psi_n(t) = \mathcal{L}^{-1}[\Psi_n(s); s \to t] = \int_{-\infty}^{t-1} (t-\eta)^{\frac{n-1}{2}}.\frac{t-\eta}{\sqrt{(t-\eta)^2-1}}\delta(\eta)d\eta = \frac{t^{\frac{n+1}{2}}}{\sqrt{t^2-1}}.$$

Finally using convolution theorem for the Laplace transform, we have the following relation

$$\psi(t) = \mathcal{L}^{-1}[\Psi_n(s)\Psi_m(s); s \to t] = \int_0^t \frac{(t-\xi)^{\frac{m+1}{2}}}{\sqrt{(t-\xi)^2-1}}\frac{\xi^{\frac{n+1}{2}}}{\sqrt{\xi^2-1}}d\xi.$$

□

2. Generalized Bessel's Equation, Bessel Functions, Hankel Transform

Let us consider the following second order differential equation with non-constant coefficients

$$x^2y'' + (1-2\alpha)xy' + [(kcx^c)^2 + \alpha^2 - \nu^2c^2]y = 0, \tag{2.1}$$

the above equation has the following solution

$$y(x) = x^{\alpha}[C_1 J_{\nu}(kx^c) + C_2 Y_{\nu}(kx^c)]. \tag{2.2}$$

We note that if $\alpha = 0, c = 1$ we obtain the Bessel equation

$$x^2 y'' + xy' + [(kx)^2 - \nu^2 c^2]y = 0, \tag{2.3}$$

with the solution as follows

$$y(x) = C_1 J_{\nu}(kx) + C_2 Y_{\nu}(kx). \tag{2.4}$$

In Eq.(2.1), if we set $\alpha = 0.5,\ c = \frac{3}{2},\ \nu = \frac{1}{3},\ k = \frac{2i}{3}$ we get

$$x^2 y'' + [(ix^{\frac{3}{2}})^2 + \frac{1}{4} - \frac{1}{4}]y = 0, \tag{2.5}$$

after simplifying we obtain

$$y'' - xy = 0, \tag{2.6}$$

the above equation is known as an Airy differential equation with the solution as below

$$y(x) = \sqrt{x}[C_1 J_{\frac{1}{3}}(\frac{2i}{3}x^{\frac{3}{2}}) + C_2 J_{-\frac{1}{3}}(\frac{2i}{3}x^{\frac{3}{2}})]. \tag{2.7}$$

At this stage using the fact that

$$J_{\nu}(ix) = e^{\frac{-i\pi\nu}{2}} I_{\nu}(x), \qquad K_{\nu}(x) = \frac{2}{\sin(\pi\nu)}[I_{-\nu}(x) - I_{\nu}(x)].$$

Where $I_{\nu}(x),\ K_{\nu}(x)$ are the modifed Bessel functions of the first and second kind respectively. Therefore, we get

$$y(x) = \sqrt{x}[C_1' I_{\frac{1}{3}}(\frac{2}{3}x^{\frac{3}{2}}) + C_2' I_{-\frac{1}{3}}(\frac{2}{3}x^{\frac{3}{2}})]. \tag{2.8}$$

In the special case $\nu = \frac{1}{3}$, we have the following relations [8,11]

$$Ai(x) = \frac{1}{\pi}\sqrt{\frac{x}{3}} K_{\frac{1}{3}}(\frac{2x^{\frac{3}{2}}}{3}) = \frac{\sqrt{x}}{3}[I_{-\frac{1}{3}}(\frac{2x^{\frac{3}{2}}}{3}) - I_{\frac{1}{3}}(\frac{2x^{\frac{3}{2}}}{3})]$$

and

$$Bi(x) = \sqrt{\frac{x}{3}}[I_{-\frac{1}{3}}(\frac{2x^{\frac{3}{2}}}{3}) + I_{\frac{1}{3}}(\frac{2x^{\frac{3}{2}}}{3})].$$

Finally, equation (2.6) has the following solution in terms of the Airy functions $Ai(x),\ Bi(x)$

$$y(x) = C_1'' Ai(x) + C_2'' Bi(x).$$

Remark. It is worth mentioning that the Airy function $Ai(x)$ is used in physics to model of the diffraction of light.

Theorem 2.1. *We have the following integral representation of the square of the Airy function*

$$Ai^2(\phi) = \frac{1}{\pi\sqrt{3}} \int_0^{+\infty} \eta J_0(2\phi\eta + \frac{2\eta^3}{3}) d\eta. \tag{2.9}$$

Note. In the literature the integral representation of the square of the Airy function is given [8,11].

Proof. Let us start with an integral representation of the product of the modified Bessel functions of order ν as follows

$$K_\nu(x)K_\nu(y) = \frac{\pi}{2\sin(\pi\nu)} \int_{\ln(\frac{y}{x})}^{+\infty} J_0(\sqrt{2xy\cosh\xi - (x^2+y^2)})\sinh(\nu\xi)d\xi,$$

by taking $\nu = \frac{1}{3},\ x = y$, we have the following relation [8]

$$K^2_{\frac{1}{3}}(x) = \frac{\pi}{2\sin(\frac{\pi}{3})} \int_0^{+\infty} J_0(x\sqrt{2\cosh\xi - 2})\sinh(\frac{\xi}{3})d\xi.$$

At this stage using the well-known identity $K_{\frac{1}{3}}(x) = \frac{\pi\sqrt{3}}{\sqrt{\phi}} Ai(\phi)$, where $x = \frac{2}{3}\phi^{\frac{3}{2}}$, therefore, we have

$$[\frac{\pi\sqrt{3}}{\sqrt{\phi}} Ai(\phi)]^2 = \frac{\pi}{\sqrt{3}} \int_0^{+\infty} J_0(\frac{2}{3}\phi^{\frac{3}{2}}\sqrt{2\cosh\xi - 2}) 2\sinh(\frac{\xi}{6})\cosh(\frac{\xi}{6})d\xi,$$

after simplifying we obtain

$$Ai^2(\phi) = \frac{\phi}{3\pi\sqrt{3}} \int_0^{+\infty} J_0[\frac{4\phi\sqrt{\phi}}{3}(3\sinh(\frac{\xi}{6}) + 4\sinh^3(\frac{\xi}{6}))] 2\sinh(\frac{\xi}{6})\cosh(\frac{\xi}{6})d\xi.$$

Let us introduce a change of variable $\sinh(\frac{\xi}{6}) = \frac{\eta}{2\sqrt{\phi}}$, then we have $\frac{1}{6}\cosh(\frac{\xi}{6})d\xi = \frac{d\eta}{2\sqrt{\phi}}$, from which we deduce that

$$Ai^2(\phi) = \frac{\phi}{3\pi\sqrt{3}} \int_0^{+\infty} J_0[\frac{4\phi\sqrt{\phi}}{3}[(\frac{3\eta}{2\sqrt{\phi}}) + 4(\frac{\eta}{2\sqrt{\phi}})^3]] 12 \frac{\eta}{2\sqrt{\phi}} \frac{d\eta}{2\sqrt{\phi}}.$$

Finally, we obtain

$$Ai^2(\phi) = \frac{1}{\pi\sqrt{3}} \int_0^{+\infty} \eta J_0(2\phi\eta + \frac{2\eta^3}{3}) d\eta. \tag{2.10}$$

Let us consider the following special cases

1. $\phi = 0$, we get

$$Ai^2(0) = \frac{1}{3^{\frac{4}{3}}\Gamma^2(\frac{2}{3})} = \frac{1}{\pi\sqrt{3}} \int_0^{+\infty} \eta J_0(\frac{2\eta^3}{3}) d\eta.$$

2. In Eq.(2.10), taking derivitive with respect to ϕ and setting $\phi = 0$, we have

$$2Ai(0)Ai'(0) = \frac{-2}{2\pi\sqrt{3}} = \frac{-2}{\pi\sqrt{3}} \int_0^{+\infty} \eta^2 J_1(\frac{2\eta^3}{3}) d\eta,$$

or

$$\int_0^{+\infty} 2\eta^2 J_1(\frac{2\eta^3}{3})d\eta = \int_0^{+\infty} J_1(\tau)d\tau = 1.$$

□

Theorem 2.2. *We have the following integral identity for the modified Bessel function of the second kind or Macdonald function*

$$\int_0^{+\infty} K_\nu(\lambda\sqrt{x^2+z^2})\frac{x^{2\beta+1}}{(x^2+z^2)^{\frac{\nu}{2}}}dx = \frac{2^\beta\Gamma(\beta+1)}{\lambda^{\beta+1}z^{\nu-(\beta+1)}}K_{\nu-(\beta+1)}(\lambda z).$$

Proof. Let us start with the left hand side, by using an integral representation for the modified Bessel function, we have

$$\int_0^{+\infty} K_\nu(\lambda\sqrt{x^2+z^2})\frac{x^{2\beta+1}}{(x^2+z^2)^{\frac{\nu}{2}}}dx =$$

$$= \int_0^{+\infty} \frac{x^{2\beta+1}}{(x^2+z^2)^{\frac{\nu}{2}}}[(\frac{\lambda(\sqrt{x^2+z^2})}{2})^\nu \int_0^{+\infty} e^{-\xi-\frac{\lambda^2(x^2+z^2)}{4\xi}}\frac{d\xi}{2\xi^{\nu+1}}]dx,$$

changing the order of integration in the double integral and after simplifying, we obtain

$$L.H.S = (\frac{\lambda}{2})^\nu \int_0^{+\infty} e^{-\xi-\frac{\lambda^2z^2}{4\xi}}[\int_0^{+\infty} x^{2\beta+1}e^{-\frac{\lambda^2x^2}{4\xi}}dx]\frac{d\xi}{2\xi^{\nu+1}}.$$

At this point let us make a change of variable $u = \frac{\lambda^2x^2}{4\xi}$ in the inner integral after simplification we obtain

$$L.H.S = \frac{1}{2}(\frac{\lambda}{2})^\nu\Gamma(\beta+1)(\frac{2}{\lambda})^{2(\beta+1)} \int_0^{+\infty} e^{-\xi-\frac{\lambda^2z^2}{4\xi}}\frac{d\xi}{2\xi^{\nu+1}}.$$

Let us rewrite the above relation as follows

$$L.H.S = \frac{1}{2}(\frac{\lambda}{2})^\nu\Gamma(\beta+1)(\frac{2}{\lambda})^{2(\beta+1)}(\frac{\lambda z}{2})^{-\nu+(\beta+1)})[(\frac{\lambda z}{2})^{\nu-(\beta+1)}(\int_0^{+\infty} e^{-\xi-\frac{\lambda^2z^2}{4\xi}}\frac{d\xi}{2\xi^{\nu+1}}].$$

But the expression in the brackets is the integral representation for the modified Bessel function $K_{\nu-(\beta+1)}(\lambda z)$, therefore we get

$$L.H.S = \frac{2^\beta\Gamma(\beta+1)}{z^{\nu-(\beta+1)}\lambda^{\beta+1}}K_{\nu-(\beta+1)}(\lambda z).$$

Let us consider the special case $\nu = 0$ then we get

$$\int_0^{+\infty} K_0(\lambda\sqrt{x^2+z^2})x^{2\beta+1}dx = \frac{2^\beta\Gamma(\beta+1)}{\lambda^{\beta+1}z^{-(\beta+1)}}K_{-(\beta+1)}(\lambda z) = \frac{2^\beta z^{(\beta+1)}\Gamma(\beta+1)}{\lambda^{\beta+1}}K_{(\beta+1)}(\lambda z).$$

Also, considering the special case $\beta = -\frac{1}{2}$ we obtain

$$\int_0^{+\infty} K_0(\lambda\sqrt{x^2+z^2})dx = \sqrt{\frac{\lambda z}{2\pi}} K_{\frac{1}{2}}(\lambda z).$$

In the above theorem, we used the fact that $K_\nu(.) = K_{-\nu}(.)$ and the well-known integral representation $K_\nu(az) = (\frac{az}{2})^\nu \int_0^{+\infty} e^{-\xi - \frac{a^2z^2}{4\xi}} \frac{d\xi}{2\xi^{\nu+1}}$, [5,8]. $\square$

Hankel Transforms

Hankel transforms arise naturally in solving boundary-value problems formulated in cylindrical coordinates. They also occur in other applications such as determining oscillations of the suspended heavy chain from one end.
We define the general Hankel transforms of order ν by

$$\mathcal{H}_\nu[\phi(r);\rho] = \int_0^{+\infty} rJ_\nu(\rho r)\phi(r)dr = \Phi(\rho). \tag{2.11}$$

The corresponding inversion formula of which takes the form

$$\mathcal{H}_\nu^{-1}[\Phi(\rho);r] = \int_0^{+\infty} \rho J_\nu(r\rho)\Phi(\rho)d\rho = \phi(r). \tag{2.12}$$

The basic requirement for the existence of the Hankel transform is that the function $\sqrt{r}f(r)$ be absolutely integrable and piecewise continuous on the positive real line. In this section we will determine the Hankel transform of certain functions and develop some of the fundamental operational properties of the Hankel transform.

Lemma 2.1. *Let us assume that $\mathcal{H}_\nu[\phi(r);\rho] = \Phi(\rho)$, then we have*

$$1. \qquad \mathcal{H}_\nu[\frac{1}{r^{\nu+1}}\frac{d}{dr}[r^{2\nu+1}\frac{d}{dr}(\frac{1}{r^\nu}\phi(r))];\rho] = -\rho^2\Phi(\rho). \tag{2.13}$$

$$2. \qquad \mathcal{H}_0[\frac{1}{r}\frac{d}{dr}[r\frac{d}{dr}(\phi(r))];\rho] = -\rho^2\Phi(\rho). \tag{2.14}$$

Proof. See [3,4,9].

Example 2.1. Show that

$$\mathcal{H}_0[\frac{1}{\sqrt{r^2+a^2}};\rho] = \frac{1}{\rho}e^{-a\rho}.$$

Proof. Let us start with the Laplace transform of the function $J_0(r\rho)$, we have

$$\mathcal{L}[J_0(r\rho);\rho\to a] = \int_0^{+\infty} e^{-a\rho}J_0(r\rho)d\rho = \frac{1}{\sqrt{a^2+r^2}}.$$

In terms of the Hankel transform of order zero we have

$$\mathcal{H}_0[\frac{e^{-a\rho}}{\rho}; \rho \to r] = \frac{1}{\sqrt{r^2 + a^2}}.$$

Inverting the above relation leads to

$$\mathcal{H}_0^{-1}[\frac{1}{\sqrt{r^2 + a^2}}; r \to \rho] = \int_0^{+\infty} \rho J_0(\rho r) \frac{1}{\sqrt{r^2 + a^2}} dr = \frac{e^{-a\rho}}{\rho}.$$

□

Lemma 2.2. *Parseval identity for the Hankel transform.*
If $\Phi(\rho)$ and $\Psi(\rho)$ are the Hankel transforms of the functions $\phi(r)$ and $\psi(r)$, respectively, then

$$\int_0^{+\infty} r\phi(r)\psi(r)dr = \int_0^{+\infty} \rho\Phi(\rho)\Psi(\rho)d\rho. \tag{2.15}$$

Proof. The integral on the right side can be rewritten as follows

$$\int_0^{+\infty} \rho\Phi(\rho)\Psi(\rho)d\rho = \int_0^{+\infty} \rho\Phi(rho)[\int_0^{+\infty} rJ_\nu(\rho r)\psi(r)dr]d\rho.$$

Changing the order of integration, we get

$$\int_0^{+\infty} \rho\Phi(\rho)\Psi(\rho)d\rho = \int_0^{+\infty} r\psi(r)[\int_0^{+\infty} \rho J_\nu(r\rho)\Phi(\rho)d\rho]dr = \int_0^{+\infty} r\psi(r)\phi(r)dr.$$

□

Lemma 2.3. *The following integral identity holds*

$$\frac{1}{2}\delta(\frac{a^2 - b^2}{4}) = \int_0^{+\infty} \rho J_\nu(a\rho)J_\nu(b\rho)d\rho. \tag{2.16}$$

Proof. Let us take $\phi(r) = \frac{1}{2}\delta(\frac{r^2-a^2}{4})$ and $\psi(r) = \frac{1}{2}\delta(\frac{r^2-b^2}{4})$. In view of the Parseval identity and using Lemma 2.4. we have

$$\int_0^{+\infty} \frac{1}{2}\delta(\frac{r^2 - a^2}{4})\frac{1}{2}\delta(\frac{r^2 - b^2}{4})rdr = \frac{1}{2}\delta(\frac{a^2 - b^2}{4}) = \int_0^{+\infty} \rho J_\nu(a\rho)J_\nu(b\rho)d\rho. \tag{2.17}$$

□

Lemma 2.4. *We have the following relations for the Hankel transform*

$$\mathcal{H}_\nu[\frac{1}{2}\delta(\frac{r^2 - a^2}{4}); \rho] = \int_0^{+\infty} rJ_\nu(\rho r)\delta(\frac{r^2 - a^2}{4})dr = J_\nu(a\rho). \tag{2.18}$$

Proof. Let us make a change of variable $\xi = \frac{r^2-a^2}{4}$ in the above integral, we get

$$\mathcal{H}_\nu[\frac{1}{2}\delta(\frac{r^2-a^2}{4});\rho] = \int_{-\frac{a^2}{4}}^{+\infty} \sqrt{4\xi+a^2}J_\nu(\rho\sqrt{4\xi+a^2})\delta(\xi)\frac{2d\xi}{\sqrt{4\xi+a^2}} = J_\nu(a\rho). \tag{2.19}$$

$\square$

3. Solution for the Time Fractional Heat Equation in Cylindrical Coordinates Via the Joint Laplace-Hankel Transform

Fractional calculus deals with the fractional integrals and derivatives of arbitrary order. It provides better models for systems having long range memory and non-local effects and it has important applications in several fields of engineering and sciences. Fractional differential equations are widely used for modeling anomalous diffusion phenomena. In this section, the author implemented the joint Laplace-Hankel transforms to construct the exact solution for the time fractional heat conduction equation. In the past three decades, considerable research work has been invested in the study of the anomalous diffusion using the time fractional equation.

Problem 3.1 Let us solve the following impulsive time fractional heat conduction equation in cylindrical coordinates

$$D_t^{c,\alpha}u = \frac{a^2}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \delta(t)\delta(r-r_0), \qquad \alpha = 0.5, \quad t>0, \quad 0<r<+\infty.$$

with the boundary conditions as follows

$$1. \quad u(r,0)=f(r), \quad 2. \quad \lim_{r\to 0}|u(r,t)|<+\infty, \quad 3. \quad \lim_{r\to+\infty} u(r,t)=0.$$

Solution. Let us define the joint Laplace-Hankel transform of order zero as follows

$$U(\rho,s) = \int_0^{+\infty} rJ_0(\rho r)[\int_0^{+\infty} e^{-st}u(r,t)dt]dr. \tag{3.1}$$

Application of the joint Laplace-Hankel transform the above equation leads to the following transformed equation with the boundary conditions as follows

$$s^\alpha U(\rho,s) + a^2\rho^2 U(\rho,s) = s^{\alpha-1}F(\rho) + r_0J_0(r_0\rho), \qquad \mathcal{H}_0[f(r);\rho] = F(\rho). \tag{3.2}$$

Solving the above equation (3.2) yields

$$U(\rho,s) = \frac{s^{\alpha-1}F(\rho)+r_0J_0(r_0\rho)}{s^\alpha+a^2\rho^2} = F(\rho)[\frac{1}{s^{1-\alpha}(s^\alpha+a^2\rho^2)}] + J_0(r_0\rho)\frac{r_0}{s^{1-\alpha}(s^\alpha+a^2\rho^2)}. \tag{3.3}$$

At this point, taking the joint inverse Laplace-Hankel transform of order zero to obtain

$$\begin{aligned} u(r,t) &= \int_0^{+\infty} \rho J_0(r\rho)F(\rho)[\mathcal{L}^{-1}[\frac{1}{s^{1-\alpha}(s^\alpha + a^2\rho^2)}]d\rho \\ &+ r_0 \int_0^{+\infty} \rho J_0(r_0\rho)J_0(r\rho)[\mathcal{L}^{-1}[\frac{1}{s^{1-\alpha}(s^\alpha + a^2\rho^2)}]d\rho. \end{aligned} \tag{3.4}$$

At this stage let us take $\alpha = 0.5$, then we have

$$\mathcal{L}^{-1}[\frac{1}{s^{1-\alpha}(s^\alpha + a^2\rho^2)}] = \mathcal{L}^{-1}[\frac{1}{\sqrt{s}(\sqrt{s} + a^2\rho^2)}] = e^{a^4\rho^4 t} Erfc(a^2\rho^2\sqrt{t}). \tag{3.5}$$

In relation (3.6), let us replace $F(\rho) = \mathcal{H}_0[f(r);\rho]$, $r_0 J_0(r_0\rho) = \mathcal{H}_0[\delta(r-r_0);\rho]$ by the following integrals

$$F(\rho) = \int_0^{+\infty} \xi J_0(\rho\xi)f(\xi)d\xi, \qquad r_0 J_0(r_0\rho) = \int_0^{+\infty} \tau J_0(\rho\tau)\delta(\tau - r_0)d\tau \tag{3.6}$$

we arrive at

$$\begin{aligned} u(r,t) = \int_0^{+\infty} \rho J_0(r\rho)e^{a^4\rho^4 t} Erfc(a^2\rho^2\sqrt{t})[\int_0^{+\infty} \xi J_0(\rho\xi)f(\xi)d\xi]d\rho + \\ + \int_0^{+\infty} \rho J_0(r\rho)e^{a^4\rho^4 t} Erfc(a^2\rho^2\sqrt{t})[\int_0^{+\infty} \tau J_0(\rho\tau)\delta(\tau - r_0)d\tau]d\rho \end{aligned} \tag{3.7}$$

By changing the order of integration we obtain the formal solution to boundary-value problem

$$\begin{aligned} u(r,t) = \int_0^{+\infty} \xi f(\xi)[\int_0^{+\infty} \rho J_0(r\rho)e^{a^4\rho^4 t} Erfc(a^2\rho^2\sqrt{t})J_0(\xi\rho)d\rho]d\xi + \\ + \int_0^{+\infty} \tau\delta(\tau - r_0)[\int_0^{+\infty} \rho J_0(r\rho)e^{a^4\rho^4 t} Erfc(a^2\rho^2\sqrt{t})J_0(\tau\rho)d\rho]d\tau. \end{aligned} \tag{3.8}$$

Note. In the above relation $Erfc(\xi) = \frac{2}{\sqrt{\pi}}\int_\xi^{+\infty} e^{-t^2}dt$.
The last step is to evaluate $u(r,0)$ as below

$$u(r,0) = \int_0^{+\infty} \xi f(\xi)[\int_0^{+\infty} \rho J_0(r\rho)J_0(\xi\rho)d\rho]d\xi + \int_0^{+\infty} \tau\delta(\tau - r_0)[\int_0^{+\infty} \rho J_0(r\rho)J_0(\tau\rho)d\rho]d\tau. \tag{3.9}$$

In view of the Lemma 2.4. the value of the inner integrals are $\frac{1}{2}\delta(\frac{r^2-\xi^2}{4})$ and $\frac{1}{2}\delta(\frac{r^2-\tau^2}{4})$ respectively, therefore

$$u(r,0) = \int_0^{+\infty} \xi f(\xi)[\frac{1}{2}\delta(\frac{r^2-\xi^2}{4})]d\xi + \int_0^{+\infty} \tau\delta(\tau - r_0)[\frac{1}{2}\delta(\frac{r^2-\tau^2}{4})]d\tau = f(r). \tag{3.10}$$

Note. In the last step we have made a change of variable $\frac{r^2-\xi^2}{4} = \eta$ in the above integral.

4. Main Result. Solution for The Time Fractional Non-Homogeneous Heat Equation in Cylindrical Coordinates via the Joint Laplace-Hankel Transform

Let us consider the following time fractional heat conduction equation a fractional generalization of the problem Ion distribution function during ion cyclotron resonance heating at the fundamental frequency [6]

$$D_t^{c,\alpha}u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \lambda u + \phi(r) + \mathcal{J}^{\alpha}h(t), \qquad 0<\alpha<1, \quad t>0, \quad 0<r<+\infty$$

with the boundary conditions as below

$$1. \quad u(r,0)=\psi(r), \quad 2. \quad \lim_{r\to 0}|u(r,t)|<+\infty, \quad 3. \quad \lim_{r\to+\infty}u(r,t)=0.$$

Note. Analytic solutions are more important than numerical solutions, because these are valid in the whole domain of definition whereas the numerical solutions are only valid at chosen points in the domain of definition.
Solution. Let us define the joint Laplace-Hankel transforms of order zero as follows

$$U(\rho,s) = \int_0^{+\infty} rJ_0(\rho r)[\int_0^{+\infty} e^{-st}u(r,t)dt]dr. \tag{4.1}$$

By applying the joint Laplace-Hankel transforms of order zero the above equation, we arrive at the following transformed equation with the boundary conditions

$$(s^\alpha+\rho^2+\lambda)U(\rho,s) = s^{\alpha-1}\Psi(\rho) + \frac{\Phi(\rho)}{s} + \frac{H(s)}{s^\alpha}. \tag{4.2}$$

Solution of the above equation (4.2) leads to

$$U(\rho,s) = \frac{\Psi(\rho)}{s^{1-\alpha}(s^\alpha+\rho^2+\lambda)} + \frac{\Phi(\rho)}{s(s^\alpha+\rho^2+\lambda)} + \frac{H(s)}{s^\alpha(s^\alpha+\rho^2+\lambda)}. \tag{4.3}$$

By taking the inverse joint Laplace-Hankel transform of order zero, we have

$$u(r,t) = \int_0^{+\infty}\rho J_0(r\rho)\Psi(\rho)[\mathcal{L}^{-1}[\frac{1}{s^{1-\alpha}(s^\alpha+\rho^2+\lambda)}]d\rho +$$

$$+\int_0^{+\infty}\rho J_0(r\rho)\Phi(\rho)[\mathcal{L}^{-1}[\frac{1}{s(s^\alpha+\rho^2+\lambda)}]d\rho + \int_0^{+\infty}\rho J_0(r\rho)[\mathcal{L}^{-1}[\frac{H(s)}{s^\alpha(s^\alpha+\rho^2+\lambda)}]d\rho. \tag{4.4}$$

In view of the Corollary 1.2. we have the following formal solution

$u(r,t) =$

$$= \frac{\sin(\pi\alpha)}{\pi\Gamma(1-\alpha)}\int_0^{+\infty}\rho J_0(r\rho)\Psi(\rho)[\int_0^t\frac{1}{(t-\eta)^{\alpha}}[\int_0^{+\infty}\frac{\xi^{\alpha}e^{-\eta\xi}d\xi}{\xi^{2\alpha}+2(\sqrt{\rho^2+\lambda})\xi^{\alpha}\cos(\pi\alpha)+\lambda+\rho^2}]d\eta]d\rho+$$

$$+\frac{\sin(\pi\alpha)}{\pi}\int_0^{+\infty}\rho J_0(r\rho)\Phi(\rho)[\int_0^t[\int_0^{+\infty}[\frac{\xi^{\alpha}e^{-\eta\xi}}{\xi^{2\alpha}+2(\sqrt{\rho^2+\lambda})\xi^{\alpha}\cos(\pi\alpha)+\lambda+\rho^2}]d\xi]d\eta]d\rho+$$

$$+\frac{\sin(\pi\alpha)}{\pi}\int_0^{+\infty}\rho J_0(r\rho)[\int_0^t\mathcal{J}^{\alpha}h(t-\eta)[\int_0^{+\infty}[\frac{\xi^{\alpha}e^{-\eta\xi}}{\xi^{2\alpha}+2(\sqrt{\rho^2+\lambda})\xi^{\alpha}\cos(\pi\alpha)+\lambda+\rho^2}]d\xi]d\eta]d\rho. \tag{4.5}$$

At this stage let us take $\alpha = 0.5$, then we obtain the solution as follows

$$u(r,t) = \frac{1}{\pi\Gamma(\frac{1}{2})}\int_0^{+\infty}\rho J_0(r\rho)\Psi(\rho)[\int_0^t\frac{1}{(t-\eta)^{\frac{1}{2}}}[\int_0^{+\infty}\frac{\sqrt{\xi}e^{-\eta\xi}d\xi}{\xi+\lambda+\rho^2}]d\eta]d\rho+$$

$$+\frac{1}{\pi}\int_0^{+\infty}\rho J_0(r\rho)\Phi(\rho)[\int_0^t[\int_0^{+\infty}[\frac{\sqrt{\xi}e^{-\eta\xi}}{\xi+\lambda+\rho^2}]d\xi]d\eta]d\rho$$

$$+\frac{1}{\pi}\int_0^{+\infty}\rho J_0(r\rho)[\int_0^t\mathcal{J}^{\alpha}h(t-\eta)[\int_0^{+\infty}[\frac{\sqrt{\xi}e^{-\eta\xi}}{\xi+\lambda+\rho^2}]d\xi]d\eta]d\rho. \tag{4.6}$$

At this point, we may use the following integral identity in order to evaluate the inner most integral [5]

$$\int_0^{+\infty}\frac{\sqrt{\xi}e^{-\eta\xi}}{\xi+(\lambda+\rho^2)}d\xi = \sqrt{\lambda+\rho^2}e^{\eta(\lambda+\rho^2)}\Gamma(-\frac{1}{2},\eta(\lambda+\rho^2)),$$

therefore we get

$$u(r,t)=\frac{1}{\pi\Gamma(\frac{1}{2})}\int_0^{+\infty}\rho J_0(r\rho)\Psi(\rho)[\int_0^t\frac{1}{(t-\eta)^{\frac{1}{2}}}[\sqrt{\lambda+\rho^2}e^{\eta(\lambda+\rho^2)}\Gamma(-\frac{1}{2},\eta(\lambda+\rho^2))]d\eta]d\rho+$$

$$+\frac{1}{\pi}\int_0^{+\infty}\rho J_0(r\rho)\Phi(\rho)[\int_0^t[\sqrt{\lambda+\rho^2}e^{\eta(\lambda+\rho^2)}\Gamma(-\frac{1}{2},\eta(\lambda+\rho^2))]d\eta]d\rho+$$

$$+\frac{1}{\pi}\int_0^{+\infty}\rho J_0(r\rho)[\int_0^t\mathcal{J}^{\alpha}h(t-\eta)[\sqrt{\lambda+\rho^2}e^{\eta(\lambda+\rho^2)}\Gamma(-\frac{1}{2},\eta(\lambda+\rho^2))]d\eta]d\rho. \tag{4.7}$$

Note. In the above relation $\Gamma(a,\xi) = \int_{\xi}^{+\infty} t^{s-1}e^{-t}dt$ stands for the incomplete gamma function.

5. Conclusion

The paper is devoted to studying and application of the joint Laplace-Hankel transform for solving time fractional heat equation in cylindrical coordinates. The main purpose of this work is to develop a method for finding analytic solutions of fractional PDEs, evaluation of certain integrals. These results should be applicable to obtaining solutions of a wide class of problems in applied mathematics, engineering and mathematical physics. The methods and techniques discussed in this article can also be applied to solve other types of the fractional partial differential equations.
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