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Asymptotic Properties of General Nonlinear Differential Equations Containing Nonconformable Fractional Derivatives

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Abstract: In this paper the author employs the nonconformable fractional derivative developed by J. E. Nápoles Valdes and his coauthors to study the asymptotic properties of solutions of a broad class of nonlinear fractional differential equations containing such a type of derivative. Sufficient conditions for the boundedness and convergence to zero of all solutions are presented.

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1. Introduction

In this paper we utilize the nonconformable fractional derivative introduced in [1] and [4] to study the asymptotic behavior of solutions to very general nonlinear fractional differential equations that are generalizations of Emden-Fowler and other types of ordinary (integer order) equations. One advantage of using this type of fractional derivative, which we will denote by N, is that if a function is α-order, $\alpha \in (0, 1]$, differentiable at a point $t_0 \in (0, \infty)$, then it is continuous at that point (see [1, Theorem 2.2]). Also, this fractional derivative obeys product and quotient rules that mimic those for ordinary (integer order) derivatives (see [1, Theorem 2.3]). But probably its most important feature is that it satitfies a chain rule like the one for integer order derivatives (see Lemma 2.5 below). This type of fractional derivative is well described in the paper [1]. We also obtain a Gronwall type inequality for this kind of fractional derivative.

2. Preliminaries and Basic Concepts

We begin with the notion of the nonconformable fractional derivative.

Definition 2.1. ([1, Definition 2.1], [5, Definition 1]) Let $f : [0, \infty) \to \mathbb{R}$. The nonconformable fractional derivative of f of order $\alpha \in (0, 1)$ is defined by

$$(N^\alpha f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon e^{t^{-\alpha}}) - f(t)}{\epsilon}$$

for all $t > 0$.

Remark. If $(N^\alpha f)(t)$ exists in some $(0, a)$ and $\lim_{t \to 0^+}(N^\alpha f)(t)$ exists, then we define $(N^\alpha f)(0) = \lim_{t \to 0^+}(N^\alpha f)(t)$.

Corresponding to the nonconformable fractional derivative, we have the nonconformable fractional integral.

Definition 2.2. ([5, Definition 2]) Let $f : [0, \infty) \to \mathbb{R}$. The nonconformable fractional integral of f of order $\alpha \in (0, 1)$ is defined by

$$({}_N J^\alpha_{t_0} f)(t) = \int_{t_0}^{t} \frac{f(s)}{e^{s^{-\alpha}}} ds.$$

In view of Definitions 2.1 and 2.2 it is obvious that the following lemma is needed.

Lemma 2.3. ([5, Theorem 3]) *If f is N^α–differentiable on (t_0, ∞) with $\alpha \in (0, 1]$, then for $t > t_0$:*

(a) If f is differentiable, ${}_N J^\alpha_{t_0}(N^\alpha f)(t) = f(t) - f(t_0)$.

(b) $N^\alpha({}_N J^\alpha_{t_0} f)(t) = f(t)$.

For convenience, we next give some properties of the nonconformable fractional derivative.

Lemma 2.4. *Let f and g be N^α differentiable, $\alpha \in (0, 1]$, at a point $t > 0$; then:*

(1) $N^\alpha(c) = 0$ for any constant $c \in \mathbb{R}$.

(2) $N^\alpha(fg)(t) = f(t)(N^\alpha g)(t) + g(t)(N^\alpha f)(t)$.

(3) $N^\alpha \left(\frac{f}{g}\right) = \dfrac{g(t)(N^\alpha f)(t) - f(t)(N^\alpha g)(t)}{g^2(t)}$.

(4) If f is differentiable (in the ordinary sense), then $(N^\alpha f)(t) = e^{t^{-\alpha}} f'(t)$.

Proof. This is parts (c)–(f) of Theorem 2.3 in [1]. □

Remark. ([1, p. 91]) If $(N^\alpha f)(t)$ exists for $t > 0$, then f is differentiable (in the ordinary sense) at t, and

$$f'(t) = e^{-t^{-\alpha}}(N^\alpha f)(t).$$

As mentioned earlier, a very important advantage that the nonconformable fractional derivative has over other fractional derivatives is the existence of a chain rule that mimics the one for ordinary (integer valued) derivatives. We state it here as the following lemma; its proof can be found in [1, Theorem 3.1].

Lemma 2.5. *Let $\alpha \in (0,1]$, g be N^α differentiable at $t > 0$, and f be differentiable at $g(t)$. Then*

$$N^\alpha(f \circ g)(t) = f'(g(t))(N^\alpha g)(t).$$

In the study of continuability, boundedness, stability, and other asymptotic properties of solutions of nonlinear differential equations, the kinetic energy of the system often appears as an integral such as $F(x) = \int_0^x f(s)ds$. It then becomes necessary to differentiate this quantity. By applying the above chain rule, we obtain,

$$N^\alpha F(x) = f(x(t))(N^\alpha x)(t).$$

Due to its importance, we formulate this as the following corollary.

Corollary 2.6. *Let $f : \mathbb{R} \to \mathbb{R}$ and define $F(x) = \int_0^x f(s)ds$. Then*

$$(N^\alpha F)(x) = f(x(t))(N^\alpha x)(t).$$

Remark. An intermediate value theorem for nonconformable derivatives can be found in [3, Theorem 4] framed in a multivariate setting, as can a multivariate chain rule [3, Theorem 8]. Similarly, there is an implicit function theorem [3, Theorem 12].

We conclude this section with a Gronwall type inequality for nonconformable fractional derivatives. Here, we let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = [0, \infty)$.

Lemma 2.7. *Let $c \in \mathbb{R}_+$ and a, $u : \mathbb{R} \to \mathbb{R}_+$. If*

$$u(t) \le c + ({}_N J^\alpha_{t_0} au)(t), \tag{2.1}$$

then

$$u(t) \le c \exp\{({}_N J^\alpha_{t_0} a)(t)\}. \tag{2.2}$$

Proof. If we let $K(t)$ denote the right hand side of (2.1), then it is easy to see that (2.1) can be rewritten as

$$\frac{N^\alpha K(t)}{K(t)} \le a(t).$$

This implies

$$\frac{K'(t)}{K(t)} \le e^{-t^{-\alpha}} a(t)$$

by Remark 2. Integrating, we have

$$\ln K(t) \le \ln K(t_0) + \int_{t_0}^t e^{-s^{-\alpha}} a(s)ds,$$

so

$$K(t) \le K(t_0) \exp \int_{t_0}^{t} e^{-s^{-\alpha}} a(s)ds.$$

Hence,

$$u(t) \le K(t) \le c \exp\{({}_N J_{t_0}^{\alpha} a)(t)\},$$

which proves (2.2). □

3. Main Results

Consider the perturbed nonlinear differential equation with nonconformable fractional derivatives

$$N^{\alpha}(a(t)N^{\alpha}x) + b(t,x,N^{\alpha}x) + q(t)f(x)g(N^{\alpha}x) = e(t,x,N^{\alpha}x), \tag{E}$$

where a, $q : \mathbb{R}_+ \to \mathbb{R}_+$, b, $e : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and f, $g : \mathbb{R} \to \mathbb{R}$ are continuous functions with $g(v) > 0$ for $v \in \mathbb{R}$.

Special cases of the left hand side of this equation include the Emden–Fowler equation ($b \equiv 0$ and $g \equiv 1$), the Liénard equation ($a \equiv 1 \equiv q$, $b(t,u,v) = b(u)v$, $g \equiv 1$), and the Rayleigh equation ($a \equiv 1 \equiv q$, $b(t,u,v) = b(v)$, $g \equiv 1$). We will make use of a variety of different conditions on the coefficient functions including:

$$|e(t,u,v)| \le r(t), \tag{3.1}$$

where $r : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function;

$$b(t,u,v)v \ge 0, \tag{3.2}$$

$$F(x) = \int_0^x f(s)ds \to \infty \text{ as } |x| \to \infty, \tag{3.3}$$

$$\frac{|v|}{g(v)} \le m + nG(v), \tag{3.4}$$

where m and n are nonnegative constants and $G(v) = \displaystyle\int_0^v \frac{sds}{g(s)}$,

$$\frac{v^2}{g(v)} \le MG(v) \quad \text{for all } v, \tag{3.5}$$

where M is a positive constant;

$$N^{\alpha}a(t) \ge 0, \tag{3.6}$$

and

$$a(t) \le A, \tag{3.7}$$

where $A > 0$ is a constant.

For any continuous function $d : [0, \infty) \to \mathbb{R}$, we set $(N^\alpha d)(t)_+ = \max\{(N^\alpha d)(t), 0\}$ and $N^\alpha d(t)_- = \max\{-(N^\alpha d)(t), 0\}$ which means that $(N^\alpha d)(t) = (N^\alpha d)(t)_+ - (N^\alpha d)(t)_-$. Also, if we let

$$b(t) = \exp\left\{-\left({}_N J^\alpha_{t_0} \frac{N^\alpha d(t)_-}{d(t)}\right)(t)\right\} \quad \text{and} \quad c(t) = \exp\left\{\left({}_N J^\alpha_{t_0} \frac{N^\alpha d(t)_+}{d(t)}\right)(t)\right\}$$

then $d(t) = d(t_0)b(t)c(t)$. Moreover, it is not hard to show that if

$$ {}_N J^\alpha_{t_0}\left(\frac{N^\alpha d(t)_-}{d(t)}\right)(\infty) < \infty, \tag{3.8}$$

then $d(t)$ is bounded from below away from 0, and if

$$ {}_N J^\alpha_{t_0}\left(\frac{N^\alpha d(t)_+}{d(t)}\right)(\infty) < \infty,$$

then then $d(t)$ is bounded from above.

In view of the above discussion, we list the following possible assumptions to be used in this paper:

$$ {}_N J^\alpha_{t_0}\left(\frac{N^\alpha a(s)_+}{a(s)}\right)(\infty) < \infty, \tag{3.9}$$

$$ {}_N J^\alpha_{t_0}\left(\frac{N^\alpha a(s)_-}{a(s)}\right)(\infty) < \infty, \tag{3.10}$$

$$ {}_N J^\alpha_{t_0}\left(\frac{N^\alpha q(s)_+}{q(s)}\right)(\infty) < \infty, \tag{3.11}$$

$$ {}_N J^\alpha_{t_0}\left(\frac{N^\alpha q(s)_-}{q(s)}\right)(\infty) < \infty. \tag{3.12}$$

For convenience, we will write equation (E) as the system

$$\begin{cases} N^\alpha x = y, \\ N^\alpha y = [-(N^\alpha(a(t))y - b(t, x, y) - q(t)f(x)g(y) + e(t, x, y)]/a(t). \end{cases} \tag{S_1}$$

Note: As long as there is no ambiguity to the meaning, in what follows we will write

$$ {}_N J^\alpha_{t_0} M(t) \quad \text{to mean} \quad ({}_N J^\alpha_{t_0} M)(t).$$

It is important to know that solutions to our problem can be defined for all time in the future, i.e., they are continuable. One such result is given in the following theorem. By interchanging some of the conditions, it is possible to obtain some variations of it.

Theorem 3.1. *Assume that $F(x)$ is bounded from below and conditions* (3.1), (3.2), (3.4) *and* (3.6) *hold. If $G(v) \to \infty$ as $|v| \to \infty$, then all solutions of system* (S_1) *and hence equation* (E) *are defined for all $t > 0$.*

Proof. Let $x(t)$ be a solution of equation (E) and $(x(t), y(t))$ be the corresponding solution of system (S_1), and assume that the solution is not continuable, i.e.,

$$\limsup_{t\to T^-}[|x(t)| + |y(t)|] = +\infty$$

for some $0 < T < \infty$ (that is, the solution has finite escape time).

Now $F(x(t)) \geq -K$ for some constant $K \geq 0$, so we define

$$V(t) = V(t, x(t), y(t)) = [F(x) + K]/a(t) + G(y)/q(t), \tag{3.13}$$

where we have suppressed some of the dependence on t.

Then, by Lemmas 2.4 and 2.5 and Corollary 2.6,

$$\begin{aligned} N^\alpha V(t) &= -[F(x) + K]N^\alpha a(t)/a^2(t) + f(x)N^\alpha x/a(t) - G(y)N^\alpha q(t)/q^2(t) \\ &\quad + \frac{y}{g(y)q(t)} N^\alpha y \\ &\leq -G(y)N^\alpha q(t)/q^2(t) + \frac{e(t,x,y)y}{g(y)q(t)a(t)} \\ &\leq -G(y)N^\alpha q(t)/q^2(t) + \frac{r(t)}{q(t)a(t)}\Big(m + nG(y)\Big). \end{aligned}$$

If we now integrate $N^\alpha V(t)$ from t_0 to T, we see that

$$\frac{G(y(t))}{q(t)} \leq V(t) \leq {}_N J^\alpha_{t_0}\left(\frac{G(y(t))}{q(t)}\left[N^\alpha q(t)_-/q(t) + \frac{nr(t)}{a(t)}\right]\right) + {}_N J^\alpha_{t_0}\left(\frac{mr(t)}{a(t)q(t)}\right) + V(t_0), \tag{3.14}$$

or

$$G(y(t))/q(t) \leq C + {}_N J^\alpha_{t_0}\left\{\frac{G(y(t))}{q(t)}\left[N^\alpha q(t)_-/q(t) + \frac{nr(t)}{a(t)}\right]\right\}$$

for some constant $C > 0$. By Lemma 2.7 we see that $G(y(t))/q(t)$ and hence $G(y(t))$ is bounded on $(0, T)$. This implies $y(t)$ is bounded on $(0, T)$ and an integration shows that $x(t)$ is bounded there as well. Therefore, the solution $(x(t), y(t))$ of (S_1) does not have finite escape time, and this proves the theorem. □

It is possible to formulate alternate versions of Theorem 3.1, for example, if $b(t, u, v) \equiv 0$, then obviously condition (3.2) is not needed; if $e(t, u, v) \equiv 0$, then (3.1) and (3.4) are not needed; if $a(t) \equiv 1$, (3.6) is not needed; and (3.6) can be dropped if we add condition (3.5). We leave the formulation and proofs of such results to the reader.

Based on Theorem 3.1 and its proof, we can formulate a number of different boundedness results. As an example, we have the following one. We will need the condition

$${}_N J^\alpha_{t_0}\left(\frac{r}{a}\right)(\infty) < \infty. \tag{3.15}$$

Theorem 3.2. *If conditions* (3.1)–(3.4), (3.6), (3.7), (3.12), *and* (3.15) *hold, then all solutions of equation* (E) *are bounded. If in addition,* $q(t) \le q_2 < \infty$ *and*

$$G(v) \to \infty \quad as \quad |v| \to \infty, \tag{3.16}$$

then all solutions of system (S_1) *are bounded.*

Proof. First observe that condition (3.3) ensures that $F(x)$ is bounded from below. Then proceeding as in the proof of Theorem 3.1, we obtain (3.14) and so

$$V(t) \le {}_N J_{t_0}^{\alpha} \left\{ V(t) \left[N^{\alpha} q(t)_- / q(t) + \frac{nr(t)}{a(t)} \right] \right\} + {}_N J_{t_0}^{\alpha} \left\{ \frac{mr(t)}{a(t)q(t)} \right\} + V(t_0).$$

An application of Gronwall's inequality (Lemma 2.7) and conditions (3.8) and (3.15) show that $V(t)$ is bounded. Hence, $[F(x)+K]/a(t)$ is bounded, and so $x(t)$ is bounded by (3.3) and (3.7).

Now $V(t)$ bounded implies $\frac{G(y(t))}{q(t)}$ is bounded, and the additional hypotheses imply that $y(t)$ is bounded. This completes the proof of the theorem. □

In order to show the versatility of the nonconformable fractional derivative, let us consider the special case of equation (E)

$$N^{\alpha}(N^{\alpha} x) + b(x) N^{\alpha} x + f(x) = 0, \tag{L}$$

i.e., the fractional Liènard equation, which we will write as the system

$$\begin{cases} N^{\alpha} x = y - B(x) \\ N^{\alpha} y = -f(x) \end{cases} \tag{S_2}$$

where $B(x) = {}_N J_{t_0}^{\alpha} b(x)$. Define

$$W(t) = W(t, x(t), y(t)) = \frac{y^2(t)}{2} + F(x).$$

Then along solutions of system (S_2), we have

$$N^{\alpha} W(t) = y N^{\alpha} y + f(x) N^{\alpha} x = -y f(x) + f(x)(y - B(x)) = -f(x) B(x).$$

Condition (3.2) implies $xB(x) \ge 0$, so if $xf(x) \ge 0$, we have $N^{\alpha} W(t) \le 0$. Thus, $W(t)$ is decreasing along solutions of (S_2). Standard Lyapunov stability theorems imply that the zero solution of (S_2) is stable. In addition, if $F(x) \to \infty$ as $|x| \to \infty$, then all solutions of (S_2) are bounded.

We indicated earlier that variations of Theorem 3.1 can be obtained by swapping some of the hypotheses. This is also the case for the boundedness result in Theorem 3.2. One such result is contained in the following theorem.

Theorem 3.3. *In addition to conditions* (3.2), (3.3), (3.5), (3.7), (3.10), *and* (3.12), *assume that*

$$|e(t,x,y)| \leq \frac{r(t)a(t)}{q(t)} \tag{3.17}$$

and

$$_N J_{t_0}^{\alpha}\left(\frac{r}{q}\right)(\infty) < \infty. \tag{3.18}$$

Then all solutions of equation (E) *are bounded. If, in addition,* $q(t) \leq q_2 < \infty$ *and* (3.16) *holds, then all solutions of system* (S_1) *are bounded.*

Proof. Since the proof will proceed along the same lines as that of Theorem 3.2, let us consider the terms arising from the differentiation of (3.13). First, we see that

$$-[F(x)+K]N^{\alpha}a(t)/a^2(t) \leq V(t)N^{\alpha}a(t)_-/a(t)$$

and

$$-G(y)N^{\alpha}q(t)/q^2(t) \leq V(t)N^{\alpha}q(t)_-/q(t).$$

Also,

$$\frac{y}{g(y)q(t)}[-yN^{\alpha}a(t)] \leq +\frac{MG(y)}{q(t)}\frac{N^{\alpha}a(t)_-}{a(t)} \leq MV(t)\frac{N^{\alpha}a(t)_-}{a(t)}.$$

Now if $|y| \leq 1$, then $\frac{|y|}{g(y)} \leq M_1$ for some $M_1 > 0$, and if $|y| \geq 1$, then $|y|/g(y) \leq |y|^2/g(y)$, so $\frac{|y|}{g(y)} \leq M_1 + |y|^2/g(y)$ for all y. In view of condition (3.5), it is easy to see that $\frac{|y|}{g(y)} \leq M_1 + MG(y)$ for all y. Also, (3.12) implies that $q(t) \geq q_1 > 0$. Hence, by (3.17),

$$\frac{ye(t,x,y)}{g(y)q(t)a(t)} \leq (M_1 + MG(y))\frac{r(t)}{q(t)} \leq \left(\frac{M_1}{q_1} + \frac{MG(y)}{q(t)}\right)\frac{r(t)}{q(t)}.$$

We then have

$$N^{\alpha}V(t) \leq V(t)\left\{(1+M)N^{\alpha}a(t)_-/a(t) + N^{\alpha}q(t)_-/q(t) + M\frac{r(t)}{q(t)}\right\} + \frac{M_1}{q_1}\frac{r(t)}{q(t)}.$$

Applying our Gronwall type inequality and the hypotheses easily completes the proof. □

Let us consider another Lyapunov (energy) type function,

$$W_1(t) = W_1(t,x(t),y(t)) = q(t)[F(x)+K]/a(t) + G(y). \tag{3.19}$$

Then,

$$\begin{aligned}
N^\alpha W_1(t) &\le W_1(t)\frac{N^\alpha\left(\frac{q(t)}{a(t)}\right)}{\frac{q(t)}{a(t)}} + \frac{N^\alpha a(t)_-}{a(t)}\left(\frac{y^2}{g(y)}\right) + \frac{e(t,x,y)y}{a(t)g(y)} \\
&\le W_1(t)\left[\frac{N^\alpha\left(\frac{q(t)}{a(t)}\right)}{\frac{q(t)}{a(t)}} + M\frac{N^\alpha a(t)_-}{a(t)}\right] + (M_1 + MG(y))\frac{r(t)}{a(t)} \\
&= W_1(t)\left[\frac{N^\alpha\left(\frac{q(t)}{a(t)}\right)}{\frac{q(t)}{a(t)}} + M\frac{N^\alpha a(t)_-}{a(t)} + M\frac{r(t)}{a(t)}\right] + M_1\frac{r(t)}{a(t)}.
\end{aligned}$$

Based on the above calculations, we can formulate the following result.

Theorem 3.4. *In addition to conditions* (3.1)–(3.3), (3.5), *and* (3.15), *assume that*

$$ {}_N J^\alpha_{t_0}\left(\frac{N^\alpha\left(\frac{q(t)}{a(t)}\right)}{\frac{q(t)}{a(t)}}\right)(\infty) < \infty \tag{3.20}$$

and

$$\frac{q(t)}{a(t)} \ge B_1 > 0 \tag{3.21}$$

for some constant B_1. *Then all solutions of equation* (E) *are bounded. If, in addition,* $q(t) \le q_2 < \infty$ *and* (3.16) *holds, then all solutions of system* (S_1) *are bounded.*

For our next boundedness theorem, we modify the Lyapunov (energy) functions we have been using and see that this leads to a different set of conditions to be satisfied. We begin by defining

$$v(t) = \exp\left\{{}_N J^\alpha_{t_0}\left(\frac{N^\alpha q(t)_-}{q(t)}\right)(t)\right\} \quad \text{and} \quad w(t) = \exp\left\{{}_N J^\alpha_{t_0}\left(\frac{N^\alpha a(t)_-}{a(t)}\right)(t)\right\}$$

and note that $v(t) \le 1$ and $w(t) \le 1$.

Theorem 3.5. *In addition to conditions* (3.1)–(3.3), (3.5), (3.7), (3.10), (3.12), *and* (3.15), *assume that*

$$y^2/g(y) \le N_1 \quad \textit{for all } y \tag{3.22}$$

and

$$ {}_N J^\alpha_{t_0}\left(\frac{r}{aq}\right)(\infty) < \infty. \tag{3.23}$$

Then all solutions of equation (E) *are bounded. If, in addition, If, in addition,* $q(t) \le q_2 < \infty$ *and* (3.16) *holds, then all solutions of system* (S_1) *are bounded.*

Proof. Define

$$W_2(t) = W_2(t, x(t), y(t)) = v(t)w(t)\left\{[F(x) + K]/a(t) + G(y)/q(t)\right\}. \tag{3.24}$$

Then,

$$N^\alpha W_2(t) \le v(t)w(t)\Bigg\{[F(x) + K]\frac{N^\alpha a(t)_-}{a^2(t)} + f(x)y/a(t) + \frac{y}{q(t)g(y)}N^\alpha y \\ -G(y)\frac{N^\alpha q(t)_-}{q^2(t)} + \big([F(x) + K]/a(t) + G(y)/q(t)\big)\left(\frac{N^\alpha q(t)_-}{q(t)} + \frac{N^\alpha a(t)_-}{a(t)}\right)\Bigg\}.$$

Condition (3.12) implies $q(t) \ge q_1 > 0$ and $v(t) \ge v_1 > 0$, and (3.10) implies $a(t) \ge a_1 > 0$ and $w(t) \ge w_1 > 0$. We also see that $|y|/g(y)$ is bounded for $|y| \le 1$ and $|y|/g(y) \le |y|^2/g(y)$ for $|y| > 1$, so from condition (3.22), $|y|/g(y) \le N_2$ for all y and some $N_2 > 0$. Hence,

$$N^\alpha W_2(t) \le W_2(t)\left[\frac{N^\alpha a(t)_-}{a(t)} + \frac{N^\alpha q(t)_-}{q(t)} + \frac{1}{v_1 w_1}\left(\frac{N^\alpha a(t)_-}{a(t)} + \frac{N^\alpha q(t)_-}{q(t)} + \frac{N_2}{q_1}\frac{N^\alpha a(t)_-}{a(t)}\right)\right] \\ + v(t)w(t)\frac{yr(t)}{g(y)a(t)q(t)}.$$

Therefore,

$$N^\alpha W_2(t) \le W_2(t)\left[\frac{N^\alpha a(t)_-}{a(t)} + \frac{N^\alpha q(t)_-}{q(t)} + \frac{1}{v_1 w_1}\left(\frac{N^\alpha a(t)_-}{a(t)} + \frac{N^\alpha q(t)_-}{q(t)} + \frac{N_2}{q_1}\frac{N^\alpha a(t)_-}{a(t)}\right)\right] \\ + \frac{N_2 r(t)}{a(t)q(t)}.$$

The remainder of the proof follows as before with an application of the Gronwall inequality and the conditions of the theorem. □

We conclude this section with the following observation. Notice that conditions (3.15), (3.18), and (3.23) do not require that the perturbation term e be small, even in the case where (3.1) holds. Many existing results on boundedness in the literature, even for those not involving fractional derivatives, require

$${}_N J_{t_0}^\alpha(r)(\infty) < \infty.$$

This is not the case with Theorems 3.2–3.5 in this paper.

4. Asymptotic Properties of Solutions

The publication of the paper by Hammett [2] in 1971 generated a great deal of interest in obtaining sufficient conditions for ensuring that nonoscillatory solutions $x(t)$ of various differential equations satisfy $\liminf_{t\to\infty} |x(t)| = 0$, and this interest continues to the present day. For the purposes of our discussion here, we classify solutions of equation (E) as follows. A solution of equation (E) is said to be *nonoscillatory* if for any $t_0 > 0$ there exists $t_1 > t_0$ such that $x(t) \neq 0$ for $t \geq t_1$. A solution of equation (E) is said to be *oscillatory* if for any $t_0 > 0$ there exist $t_1 > t_0$ and $t_2 > t_0$, with $x(t_1) > 0$ and $x(t_2) < 0$. A solution will be said to be a *Z-type* solution if it has arbitrarily large zeros but is eventually nonnegative or nonpositive. It turns out that asymptotic properties of nonoscillatory solutions often hold for the Z-type solutions as well.

We begin with two results that give sufficient conditions for bounded nonoscillatory and Z-type solutions to satisfy $\liminf_{t\to\infty} |x(t)| = 0$. This is followed by four theorems ensuring that all solutions of equation (E) converge to zero.

In what follows we will assume that

$$xf(x) > 0 \quad \text{if} \quad x \neq 0 \tag{4.1}$$

and that $f(x)$ is bounded away from 0 if x is bounded away from 0.

This means that the constant K appearing in the Lyapunov type functions (3.13), (3.19), and (3.24) can be chosen to be 0. In addition, we will use the conditions:

if u is bounded, there exists a continuous function $k_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|b(t, u, v)| \leq k_1(t)g(v), \tag{4.2}$$

$$g(v) \geq C \quad \text{for some constant } C > 0, \tag{4.3}$$

$${}_N J^{\alpha}_{t_0}(q)(\infty) = \infty, \tag{4.4}$$

$$\frac{k_1(t)}{q(t)} \to 0 \quad \text{and} \quad \frac{r(t)}{q(t)} \to 0 \text{ as } t \to \infty, \tag{4.5}$$

$${}_N J^{\alpha}_{t_0}\left(\frac{1}{a}\right)(\infty) = \infty, \tag{4.6}$$

$$a(t)k_1(t) \to 0 \quad \text{and} \quad a(t)r(t) \to 0 \;\; \text{as} \;\; t \to \infty. \tag{4.7}$$

Theorem 4.1. *Assume conditions* (3.1) *and* (4.1)–(4.6) *hold. If $x(t)$ is a bounded nonoscillatory or Z-type solution of* (E)*, then* $\liminf_{t\to\infty} |x(t)| = 0$.

Proof. If $x(t)$ is a Z-type solution, the conclusion obviously holds, so let $x(t)$ is a bounded nonoscillatory solution of (E), say $0 < x(t) < c_1$ for $t \geq t_0 > 0$ and some $c_1 > 0$. The proof in case $x(t)$ is eventually negative is similar. If $\liminf_{t\to\infty} x(t) \neq 0$, then there exists $t_1 \geq t_0$ and $c_2 > 0$ so that $x(t) \geq c_2$ for $t \geq t_1$. Thus, $f(x(t)) > c_3 > 0$ for $t \geq t_1$ for some c_3 by (4.1).

From equation (E) we have

$$\begin{aligned} N^{\alpha}(a(t)N^{\alpha}x)/g(N^{\alpha}x) &\leq -b(t, x, N^{\alpha}x)/g(N^{\alpha}x) - q(t)f(x) + e(t, x, N^{\alpha}x)g(N^{\alpha}x) \\ &\leq k_1(t) - q(t)c_3 + r(t)/C \\ &\leq q(t)[k_1(t)/q(t) - c_3 + r(t)/q(t)]. \end{aligned}$$

Since $k_1(t)/q(t) \to 0$ and $r(t)/q(t) \to 0$ as $t \to \infty$, we can choose $t_2 > t_1$ such that

$$N^{\alpha}(a(t)N^{\alpha}x)/g(N^{\alpha}x) \le -(c_3/3)q(t)$$

for $t \ge t_2$.

Integrating and applying condition (4.4) shows that $a(t)N^{\alpha}x$ is eventually negative, and this fact together with condition (4.6) shows that $x(t)$ is eventually negative, which is a contradiction. Therefore, $\liminf_{t\to\infty} x(t) = 0$. □

We also have the companion result.

Theorem 4.2. *Assume that conditions* (3.1), (4.1)–(4.3), (4.6), *and* (4.7) *hold, and* $a(t)q(t) \ge B_2$ *for some* $B_2 > 0$. *If* $x(t)$ *is a bounded nonoscillatory or* Z*-type solution of* (E), *then* $\liminf_{t\to\infty} |x(t)| = 0$.

Proof. Proceeding as in the proof of Lemma 4.1, we arrive at

$$\begin{aligned} N^{\alpha}(a(t)N^{\alpha}x)/g(N^{\alpha}x) &\le k_1(t) - q(t)c_3 + r(t)/C \\ &\le \frac{1}{a(t)}[a(t)k_1(t) - a(t)q(t)c_3 + a(t)r(t)/C]. \end{aligned}$$

Condition (4.7) implies there exits $T > 0$ such that

$$N^{\alpha}(a(t)N^{\alpha}x)/g(N^{\alpha}x) \le \frac{B_2c_3}{2}$$

for $t \ge T$. The remainder of the proof is similar to that of Theorem 4.1 □

Our first theorem guaranteeing that all solutions converge to zero is built upon Theorem 3.2.

Theorem 4.3. *If conditions* (3.1)–(3.4), (3.6), (3.7), (3.12), *and* (3.15) *hold, then every solution of* (E) *converges to zero as* $t \to \infty$.

Proof. Let $x(t)$ be solution of (E). By Theorem 3.2, $x(t)$ is bounded. Define $V(t)$ as in the proof of Theorem 3.2 (see (3.13) in the proof of Theorem 3.1) taking (4.1) into account. Differentiating, we obtain

$$N^{\alpha}V(t) \le \left\{V(t)\left[N^{\alpha}q(t)_-/q(t) + \frac{nr(t)}{a(t)}\right]\right\} + \frac{mr(t)}{a(t)q(t)}. \tag{4.8}$$

From the proof of Theorem 3.2, we have that $V(t)$ is bounded, say $V(t) \le K_1$ for some $K_1 > 0$. Let $\epsilon > 0$ be given. By conditions (3.12) and (3.15), we can choose $T_\epsilon > t_0$ such that

$${}_NJ^{\alpha}_{T_\epsilon}\left(\frac{N^{\alpha}q(s)_-}{q(s)}\right)(t) < \frac{\epsilon}{4K_1} \quad \text{and} \quad {}_NJ^{\alpha}_{T_\epsilon}\left(\frac{r}{a}\right)(t) < \min\left\{\frac{q_1\epsilon}{4m}, \frac{\epsilon}{4nK_1}\right\}$$

for $t \ge T_\epsilon$. Then, an integration of (4.8) shows that $V(t) \le \epsilon$ for $t \ge T_\epsilon$, that is,

$$\frac{F(x(t))}{A} \le \frac{F(x(t))}{a(t)} \le V(t) \to 0$$

as $t \to \infty$, which implies $x(t) \to 0$ as $t \to \infty$. □

Our next theorem is based on Theorem 3.3.

Theorem 4.4. *Let conditions* (3.2)*,* (3.3)*,* (3.5)*,* (3.7)*,* (3.10)*, and* (3.12)*,* (3.17) *and* (3.18) *hold. Then then every solution of* (E) *converges to zero as* $t \to \infty$*.*

Proof. Let $x(t)$ be a solution of (E); it is bounded by Theorem 3.3. Define $V(t)$ as used in the proof of Theorem 4.3. Differentiating, we obtain

$$N^\alpha V(t) \le V(t)\left\{(1+M)N^\alpha a(t)_-/a(t) + N^\alpha q(t)_-/q(t) + M\frac{r(t)}{q(t)}\right\} + \frac{M_1}{q_1}\frac{r(t)}{q(t)}.$$

Again $V(t)$ is bounded, say $V(t) \le K_2$ for some $K_2 > 0$. Let $\epsilon > 0$. We then find $T_1 > t_0$ so that

$$_N J^\alpha_{T_1}\left(\frac{N^\alpha a(s)_-}{a(s)}\right)(t) < \frac{\epsilon}{4(1+M)K_1}, \quad {_N J^\alpha_{T_1}}\left(\frac{N^\alpha q(s)_-}{q(s)}\right)(t) < \frac{\epsilon}{4K_1}$$

and

$$_N J^\alpha_{T_1}\left(\frac{r}{q}\right)(t) < \min\left\{\frac{\epsilon}{4MK_1}, \frac{q_1\epsilon}{K_1 M_1}\right\}$$

for $t \ge T_1$. The remainder of the proof follows as before. □

Corresponding to the boundedness result in Theorem 3.4 we have the following theorem.

Theorem 4.5. *Let conditions* (3.1)*–*(3.3)*,* (3.5)*,* (3.15)*,* (3.20) *and* (3.21) *hold. Then any solution* $x(t)$ *of equation* (E) *satisfies* $x(t) \to 0$ *as* $t \to \infty$*.*

Proof. Let $x(t)$ be a solution of (E) and define $W_1(t)$ by

$$W_1(t) = W_1(t, x(t), y(t)) = q(t)F(x)/a(t) + G(y).$$

We then have

$$N^\alpha W_1(t) = W_1(t)\left[\frac{N^\alpha\left(\frac{q(t)}{a(t)}\right)}{\frac{q(t)}{a(t)}} + M\frac{N^\alpha a(t)_-}{a(t)} + M\frac{r(t)}{a(t)}\right] + M_1\frac{r(t)}{a(t)}.$$

The boundedness of W_1 follows from the conditions in the theorem. Denote this fact by $W_1(t) \le K_3$ for all $t > t_0$ and let $\epsilon > 0$ be given. Our conditions allow us to choose $T_2 > t_0$ such that

$$_N J^\alpha_{T_2}\left(\frac{N^\alpha\left(\frac{q(t)}{a(t)}\right)}{\frac{q(t)}{a(t)}}\right)(t) < \frac{\epsilon}{4K_3} \quad {_N J^\alpha_{T_2}}\left(\frac{N^\alpha a(t)_-}{a(t)}\right)(t) < \frac{\epsilon}{4MK_3}$$

and

$$_N J^\alpha_{T_2}\left(\frac{r(t)}{a(t)}\right)(t) < \frac{\epsilon}{4K_3(M+M_1)}$$

for $t \ge T_2$. The remainder of the proof proceeds as before. □

Based on Theorem 3.5 we have our last result in this paper.

Theorem 4.6. *Let conditions* (3.1)–(3.3), (3.5), (3.7), (3.10), (3.12), *and* (3.15) (3.22) *and* (3.23) *hold. Then every solution* $x(t)$ *of equation* (E) *converges to* 0 *as* $t \to \infty$.

Proof. With $W_2(t)$ defined as in the proof of Theorem 3.5, we find that

$$N^\alpha W_2(t) \le W_2(t)\left[\frac{N^\alpha a(t)_-}{a(t)} + \frac{N^\alpha q(t)_-}{q(t)} + \frac{1}{v_1 w_1}\left(\frac{N^\alpha a(t)_-}{a(t)} + \frac{N^\alpha q(t)_-}{q(t)}\right)\right] + \frac{N_2 r(t)}{a(t)q(t)}$$

and $W_2(t) \le K_4$ for $t \ge t_0$.

For a given $\epsilon > 0$, we choose $T_3 > t_0$ with

$$_N J^\alpha_{T_2}\left(\frac{N^\alpha a(t)_-}{a(t)}\right)(t) < \frac{\epsilon}{K_4(1+\frac{1}{v_1 w_1})}, \quad _N J^\alpha_{T_2}\left(\frac{N^\alpha q(t)_-}{q(t)}\right)(t) < \frac{\epsilon}{K_4(1+\frac{1}{v_1 w_1})}$$

and

$$_N J^\alpha_{T_2}\left(\frac{r(t)_-}{a(t)q(t)}\right)(t) < \frac{\epsilon}{4}$$

for all $t \ge T_2$. The remainder of the proof is straightforward and is left to the reader. □

In conclusion, we wish to point out that all the results in this section are new for fractional differential equations of any type. Also, we remark that it would be interesting to apply this definition of a nonconformable fractional derivative to equations on time scales.

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