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On Some Metric in the Family of Compact Convex Sets

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ABSTRACT: We define the Demyanov metric and new metric and compare with the Hausdorff and Vitale metrics. Vitale compared the Hausdorff metric ρ_H and Vitale metric ρ_V. We proved that main metric ρ_{LV} is equivalence with ρ_H metric and that the family of nonempty, convex, compact sets and the ρ_{LV} metric is the complete space.

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1. Introduction

In the convex sets space metrics has a crucial role which we use for approximation of convex sets, optimization, multifunction theory, control theory etc.

The well-known a Hausdorff metric is widely applied. In some situations the Hausdorff metric is not fine enough to capture some changes in sets which may be crucial. If we rotate a polytope then the Hausdorff distance is small but their faces will not be parallel. Diamond et al in [2] reformulated the Demyanov metric and showed that the Demyanov metric majorizes the Hausdorff metric but this metrics are not equivalent for the of compact, convex sets. In 1985 R.A. Vitale give a metric similar to metric use in the L^p function space and give relation from the Hausdorff metric.

2. Basic notations and preliminaries

We introduce some notation. By $\mathcal{K}^d$ will stand for the space of nonempty, compact, convex subsets of $\mathbb{R}^d$. To each $A \in \mathcal{K}^d$ we assign a support function in the direction v: $p_A(v) = sup_{a\in A} < a, v >$ where $< \cdot, \cdot >$ is a scalar product and v is a vector the unit sphere S^{d-1}.For bounded A this is a convex, positively homogenous functional on $\mathbb{R}^d$.

By $A(v) = \{a \in A :< a, v >= p_A(v)\}$ we define the face of set $A \in \mathbb{K}^d$ in the direction v.

Definition 2.1. The point $e \in A$ is exposed point of a set A if exist a vector $v \in S^{d-1}$ such that $A(v) = e$.

Let $T_A = \{v \in \mathbb{R}^d : A(v) \ is \ singleton\}$. The set T_A is a set of full Lebesque measure in $\mathbb{R}^d$. The set $\mathbb{R}^d \setminus T_A$ has always measure 0 and so for any two $A, B \in \mathbb{K}^d$ the complement of $T_A \cap T_B$ has also measure 0.

As aritmetic operations in $\mathcal{K}^d$ we use the classical ones, namely the Minkowski sum and scalar multiplication:

$$A + b = \{a + b : a \in A, \ b \in B\} \text{ for } A, B \in \mathcal{K}^d,$$
$$\lambda A = \{\lambda a : a \in A\} \text{ for } \lambda \in \mathbb{R}^d, \ A \in \mathcal{K}^d.$$

3. The Hausdorff and Demyanov metric

Definition 3.1. Let $A, B \in \mathcal{K}^d$. The Hausdorff metric is defined by

$$\rho_H(A, B) = \max\{e(A, B), e(B, A)\}$$

where $e(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|$, ($\|\cdot\|$ is Euclidean norm in $\mathbb{R}^d$).

We use this paper the following definition the Hausdorff distance

$$\rho_H(A, B) = \sup_{v\in S^{d-1}} |p_A(v) - p_B(v)|.$$

We recall a definition the stricly convex set.

Definition 3.2. The set $A \in \mathcal{K}^d$ we call stricly convex if for all $v \in \mathbb{R}^d$, $A(v)$ is singleton.
By $\bar{\mathcal{K}}^d$ be shall denote the family of nonempty, compact and stricly convex sets.

The following examples a shows that the space $(\bar{\mathcal{K}}^d, \rho_H)$ is not complete.

Example 3.3. We consider the following sequence sets in $\bar{\mathcal{K}}^2$

$$A_n = \{(x_1, x_2) : x_1^2 + 2^n x_2^2 \leq 1\}.$$

Then $\rho_H(A_n, A_m) = |\frac{1}{2^n} - \frac{1}{2^m}|$ and is so the Cauchy sequence converges to a set $A = \{(x_1, x_2) : -1 \leq x_1 \leq 1, \ x_2 = 0\}$ which is not stricly convex.

Now we define the Demyanov metric has been introduce earlier by Pliś [4].

Definition 3.4. Let $A, B \in \mathcal{K}^d$. The Demyanov distance we define

$$\rho_D(A,B) = \sup\{\|A(v) - B(v)\| : v \in T_A \cap T_B\}.$$

The triangle inequality and the symmetricity are obvious. To prove that defines a metric we remark that $A = clco\{A(v) : v \in T_A\}$,clco stands here for the closed, convex hull of a set.This equality is a consequence that every compact, convex set is the closed, convex hull of the set of its extreme points and with the Straszewicz theorem give that the set of extreme points of a set in $\mathcal{K}^d$ is contained in the closure of the set of its exposed points. So if $\rho_D(A,B) = 0$ then the boundaries of A and B coincide and $A = B$.

Use the inequality $|p_A(v) - p_B(v)| \leq \|A(v) - B(v)\|$ for all $v \in S^{d-1}$ we have that $\rho_H(A,B) \leq \rho_D(A,B)$.

The following example ilustrate that the Hausdorff metric not respond on of rotation the sets.

Example 3.5. Let be the family sets from $\mathcal{K}^2$

$$A_x = clco\{(0,0), (cosx, sinx)\}$$

where $x \in < 0, 2\pi >$. We find the Hausdorff distance

$$\rho_H(A_x, A_y) = \sqrt{(cosx - cosy)^2 + (sinx - siny)^2} = \sqrt{2(1 - cos(x-y))} =$$
$$sin|x-y| \leq |x-y|.$$

So we have that if $x \to y$ then $\rho_H(A_x, A_y) \to 0$. Now we find the Demyanov distance for the sets A_x and A_y. Fix z such that $\frac{\pi}{2} + x < z < \frac{\pi}{2} + y$. Then $A_x(z) = (0,0)$ and $A_y(z) = (cosy, siny)$ so $\rho_D(A_x, A_y) = 1$.

T.Rzeżuchowski in([5] Theorem 2.2) prove that in $\bar{\mathcal{K}^d}$ the Hausdorff metric and the Demyanov metric are equivalent. T.Rzeżuchowski prove the following theorem.

Theorem 3.6. *The metrics ρ_H and ρ_D are equivalent in $\bar{\mathcal{K}^d}$, the metric space $(\bar{\mathcal{K}^d}, \rho_H)$ is not complete and the space $(\bar{\mathcal{K}^d}, \rho_D)$ is complete.*

4. The Vitale metric

In 1985 R.Vitale in [7] defined a new metric in $\mathcal{K}^d$ as follows

$$\rho_V(A,B) = (\int_{S^{d-1}} |p_A(v) - p_B(v)|^p) d\mu(v))^{\frac{1}{p}} \text{ for } 1 \leq p < \infty$$

where $\mu(\cdot)$ is Lebesque unit measure on S^{d-1}.

The inequality $|p_A(v) - p_B(v)| \leq \sup_{v \in S^{d-1}} |p_A(v) - p_B(v)|$ is true for all $v \in S^{d-1}$ which implies immediately that $\rho_V(A,B) \leq \rho_H(A,B)$.

In [7] Vitale showing that if we have the sequence sets A_n from $\mathcal{K}^d$ and $A \in \mathcal{K}^d$ then $\rho_V(A_n, A) \to 0 \Leftrightarrow \rho_H(A_n, A) \to 0$ for all $1 \leq p < \infty$.
This fact imply the following theorem (Theorem 3 in [7]).

Theorem 4.1. *All of the ρ_V metrics, $1 \leq p < \infty$, induce the same topology on $\mathcal{K}^d$ and yield complete metric spaces in which closed, bounded sets are compact.*

5. Main result

We introduce now the new metric

$$\rho_{LV}(A, B) = (\int_{T_A \cap T_B} \|A(v) - B(v)\|^p d\mu(v))^{\frac{1}{p}}.$$

Obviously we have the inequality $\rho_{LV}(A, B) \leq \rho_D(A, B)$ and

$$\rho_V(A, B) \leq \rho_{LV}(A, B).$$

Now we consider the example which a showed that ρ_D and ρ_{LV} metrics are not equivalent.

Example 5.1. Let be the sequence sets from $\mathcal{K}^2$:

$$A_n = clco\{(-1, 1), (1, -1), (1, 1 + \frac{1}{n}), (-1, 1)\} \text{ and } A \in \mathcal{K}^2$$

where

$$A = clco\{(-1, 1), (1, -1), (1, 1), (-1, 1)\}.$$

Then the metrics are:

$$\rho_H(A_n, A) = \frac{1}{n}, \qquad \rho_V(A_n, A) = \frac{1}{n}(\frac{\pi}{2} + arctg\frac{1}{n}), \qquad \rho_D(A_n, A) = \sqrt{4 + \frac{1}{n^2}}$$

and main metric

$$\rho_{LV}(A_n, A) = \sqrt{4 + \frac{1}{n^2}} arctg\frac{1}{n}.$$

We have that $\rho_H(A_n, A) \to 0$ and $\rho_V(A_n, A) \to 0$ and $\rho_{LV}(A_n, A) \to 0$.
The Vitale result in [7] and the inequality $\rho_V(A, B) \leq \rho_{LV}(A, B)$ give that

$$\rho_{LV}(A_n, A) \to 0 => \rho_V(A_n, A) \to 0 \implies \rho_H(A_n, A) \to 0.$$

Now we showing the inverse implication.

Theorem 5.2. *Let $A \in \mathcal{K}^d$ and the sequence sets $A_n \in \mathcal{K}^d$ be such that $\rho_H(A_n, A) \to 0$. Then for all $v \in T = T_A \cap \bigcap_{n=1}^{\infty} T_{A_n}$, $\|A_n(v) - A(v)\| \to 0$.*

Proof. Fix the $v \in T$ where $T = T_A \cap \bigcap_{n=1}^{\infty} A_n$ and we assume that

$$\|A_n(v) - A(v)\|$$

not converges to 0. Exist $\alpha > 0$ and the subsequence of the sequence A_n, denoted again by A_n for which $\|y_n - A(v)\| > \alpha > 0$ for $y_n \in A_n(v)$.
Let $y_n \to y_0$, then $\|y_0 - A(v)\| \geq \alpha$. Because $A(v)$ is the exposed point then

$$< y_0, v > - < A(v), v >> 2\alpha.$$

The following condition with assume the theorem $\rho_H(A_n, A) \to 0$ imply that

$$\sup_{v \in S^{d-1}} |p_{A_n}(v) - p_A(v)| \to 0.$$

For sufficiently large n and for $\epsilon = \alpha$ this the condition give that

$$< y_n, v > - < A(v), v > \leq \epsilon.$$

Because $y_n \to y_0$ we have that

$$2\epsilon << y_0, v > - < A(v), v > \leq \epsilon.$$

This contradiction shows that $\|A_n(v) - A(v)\| \to 0$. □

The Theorem 5.2 implies:

Corollary 5.3. *Let be $A \in \mathcal{K}^d$ and the sequence sets A_n be such that*

$$T = T_A \cap \bigcap_{n=1}^{\infty} T_{A_n}.$$

Then the metrics ρ_H and ρ_{LV} are equivalent.

This result showed the following corollary:

Corollary 5.4. *The space $(\mathcal{K}^d, \rho_{LV})$ is complete.*

6. Summary

We can use the metrics to characterization of a set-valued Lipschitz map by uniformly the Lipschitz selections in the cases for ρ_V the Lipschitz maps with the convex, compact images or the ρ_{LV} Lipschitz maps with the convex, compact images.

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