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On a Unified Form of Fractional Volterra-type Integro-Differential Equations and its applications

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Abstract: In the present article, an extension for the family of Volterra-type integro-differential equations, involving a generalization of Hilfer fractional derivative with the Lorenzo-Hartley's G-function (LHGF) in the kernel, is proposed. A compact and computable solution of the considered family of integro-differential equations is established in terms of an infinite series of LHGF. Further, certain known and new special cases of the proposed family are also established. Furthermore, some examples of the integro-differential equation are also discussed. Moreover, from the application point of view, generalized fractional free-electron laser equations involving the Caputo and the Riemann-Liouville fractional derivatives are also determined. Finally, the graphical illustrations for the solutions of the studied generalized fractional free-electron laser equations are demonstrated.

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1. Introduction

The study of fractional calculus (FC) is gaining popularity in the scientific community. It is applied to analyze several complicated phenomena in applied sciences. Several fractional-order models are explored in the recent past that characterize the multi-faceted behavior of a number of systems with complex dynamics. As an emerging area the subject has a wide variety of applications in different fields such as Medical Sciences, Space Sciences, Statistical mechanics, Control systems, Nuclear-physics, Thermal power, Finance and Material sciences, etcetera. Substantial development for the profound understanding of fractional calculus is noticed in the last few decades. For some recent and new real-world applications of fractional calculus, we refer to Sun at al. [62]. To review and insightful study of different concepts of fractional calculus we refer to standard monographs such as [6, 7, 15, 24, 30, 32, 37, 39, 41, 42, 44, 48, 54].

Special functions (SFs) are widely used in mathematics [8]. Mathematicians and applied scientists have made lots of efforts for the development of SFs while seeking an exhaustive and unified theory for the subject matter. In most of the cases, classical SFs are emerged as solutions of ordinary differential equations/partial differential equations and represented in terms of series, or integrals, or in both [51, 52]. Several classical SFs are useful in applied analysis and may be represented as particular cases of generalized special functions (GSFs), such as Fox's H-functions, Meijer's G-functions and generalized hypergeometric functions ${}_pF_q$, etcetera (we may call them generalized classical functions). For a more detailed description of classical SFs, we refer to classical monographs [1, 51, 52].

From the available corpus of classical SFs some may be referred as Special functions for fractional calculus (SFs for FC) [32]. Most often, Special functions for fractional calculus appear in the solution of arbitrary order differential equation or may arise during modelling of complex physical systems, see [22, 29, 34, 38, 50] etcetera. FC, in general, consists of differentiation and integration of arbitrary order and involves differential and integral operator of fractional order. The development of the theory of fractional calculus is largely dependent on the development of functions for the fractional calculus [35]. Thus, one may expect that exploration about generalized functions for fractional calculus may contribute towards the establishment of a unified theory of fractional calculus. We believe that such generalizations of fractional calculus may also provide a coherent methodology for analysis and applications. Generalized fractional integral and derivative operators are generally introduced by the suitable choice of functions that appeared in the kernel by more generalized functions, particularly for more details we refer [16, 18, 19, 20, 21, 25, 26, 27, 28, 29, 49, 50, 56, 60, 61, 63]. One can believe that the future growth in the theory of fractional calculus as the generalized fractional calculus would be an outcome of the manifestation of generalized special functions in different branches of science.

Fractional-order integro-differential equations are observed frequently in modelling and analysis of physical systems, see [3, 13, 14, 55, 56]. For more background, we refer to [43] and references therein. The present paper is about the applications of generalized fractional operators and generalized functions for fractional calculus to determine a unification of several fractional-order integro-differential equations that arise in applied sciences. The work presented in this paper is inspired by the remarkable contributions of other researchers (see [3, 4, 9, 10, 13, 14, 28, 43, 55]).

We present a brief description of different classical and novel fractional calculus operators and introduce the Lorenzo-Hartley's G-function (say LHGF) in the current section. In Section 2, we propose a unification for family of fractional-ordered integro-differential equations including a generalized fractional function in the kernel and a generalized fractional derivative operator (i.e., the Hilfer-Prabhakar derivative). Next, we investigate the convergence of the obtained solution for further computational requirements. Further, some of the corollaries of Theorem 2.1 are derived in the next Section 3. For the applications of the derived unification, two examples are discussed in Section 4. Furthermore, in Section 5, solutions for two generalized fractional free-electron laser equations, involving Caputo and Riemann-Liouville derivatives respectively, are determined. Moreover, some graphical illustrations for the considered generalized fractional free-electron laser equations are demonstrated in the same Section 5. Finally, in Section 6 we present some concluding remarks.

1.1. Riemann-Liouville fractional-order derivative

If $h(t)$, where $-\infty \leq a < t < b \leq \infty$, is locally integrable real-valued function in $\mathcal{L}^1[a,b]$, then the $\mu^{th}(\mu \in \mathbb{C})$ order right-sided Riemann-Liouville fractional integral of $h(t)$ is denoted by ${}^{\mathsf{RL}}\mathbf{I}^{\mu}_{a+}h$ and defined as [42, 48, 54]:

$$({}^{\mathsf{RL}}\mathbf{I}^{\mu}_{a+}h)(t) = \frac{1}{\Gamma(\mu)}\int_a^t \frac{h(u)}{(t-u)^{1-\mu}}du = (h * \mathcal{F}_\mu)(t), \tag{1}$$

with the condition that $(t > 0; \mathfrak{Re}(\mu) > 0)$. The expression $\mathcal{F}_\mu(t)$ is given by $\mathcal{F}_\mu(t) = \dfrac{t^{\mu-1}}{\Gamma(\mu)}$.

If $h(t) \in \mathcal{L}^1[a,b]$, where $-\infty < a < t < b < \infty$ and $h * \mathcal{F}_{m-\mu} \in \mathcal{W}^{m,1}[a,b]$, $m = \lceil \mu \rceil$, $\mu > 0$, where $\lceil \cdot \rceil$ is the least integer function. Also, $\mathcal{W}^{m,1}[a,b]$ is used to denote the Sobolev space defined as:

$$\mathcal{W}^{m,1}[a,b] = \Big\{ h(t) \in \mathcal{L}^1[a,b] : \frac{d^m}{dt^m}h(t) \in \mathcal{L}^1[a,b] \Big\}. \tag{2}$$

The classical Riemann-Liouville right-sided fractional derivative of order μ ($\mu \in \mathbb{C}$, $\mathfrak{Re}(\mu) > 0$) is defined as:

$$({}^{\mathsf{RL}}\mathbf{D}^{\mu}_{a+}h)(t) = \Big(\frac{d}{dt}\Big)^m \Big(({}^{\mathsf{RL}}\mathbf{I}^{m-\mu}_{a+}h)(t)\Big) = \frac{1}{\Gamma(m-\mu)}\frac{d^m}{dt^m}\int_a^t (t-u)^{m-\mu-1}h(u)du, \tag{3}$$

with $m = -[-\mathfrak{Re}(\mu)]$, where $[\cdot]$ denotes the integral part of the argument, i.e.

$$m = \begin{cases} [\mathfrak{Re}(\mu)] + 1 & \text{for } \mu \notin \mathbb{N}_0, \\ \mu & \text{for } \mu \in \mathbb{N}_0. \end{cases} \tag{4}$$

Particularly for $\mu = m \in \mathbb{N}_0$, we write

$$({}^{\mathsf{RL}}\mathbf{D}^{\mu}_{a+}h)(t) = h^{(m)}(t), \tag{5}$$

where $h^{(m)}(t)$ is the standard m^{th} order derivative of the function $h(t)$.

If $\mathcal{AC}[a,b]$ is the space of absolutely continuous functions and $h(t)$ be the real-valued functions with continuous derivative up to order $(m-1)$ on the interval $[a,b]$ such that $h^{m-1}(t) \in \mathcal{AC}[a,b]$, we say that function $h(t) \in \mathcal{AC}^m[a,b] (m \in \mathbb{N})$. The space $\mathcal{AC}^m[a,b]$ of real-valued function is given as:

$$\mathcal{AC}^m[a,b] = \Big\{ h : [a,b] \to \mathbb{R} : \frac{d^{m-1}}{dt^{m-1}} h(t) \in \mathcal{AC}[a,b] \Big\}. \tag{6}$$

1.2. Caputo fractional-order derivative

The Caputo fractional derivative of a function $h(t)$, denoted by ${}^{\mathsf{C}}\mathbf{D}^{\mu}_{a+}h(t)$, has a close connection with Riemann-Liouville fractional derivative ${}^{\mathsf{RL}}\mathbf{D}^{\mu}_{a+}h(t)$ (see [15, 39, 41, 42]).

If $h(t) \in \mathcal{AC}^m[a,b]$, $\mu \in \mathbb{C}$ $(\mathfrak{Re}(\mu) > 0), m = \lceil \mu \rceil$ then the right-sided μ^{th} order Caputo fractional derivative of $h(t)$ is defined as:

$${}^{\mathsf{C}}\mathbf{D}^{\mu}_{a+}h(t) = \Big({}^{\mathsf{RL}}\mathbf{I}^{m-\mu}_{a+} \frac{d^m}{dt^m} h \Big)(t) = \frac{1}{\Gamma(m-\mu)} \int_a^t (t-u)^{m-\mu-1} \frac{d^m}{du^m} h(u) du. \tag{7}$$

The study of generalized fractional-order derivatives, being part of the investigation of several researchers [19, 24, 27, 28, 29, 31, 34, 40, 55, 56, 59, 60, 61, 63], are of great need as such generalized fractional derivatives play a vital role in the justification of various phenomena in different complex systems. Now we consider some of the popular generalizations of the above-defined classical fractional derivatives.

1.3. Hilfer derivative

If $h(t) \in \mathcal{L}^1[a,b], h * \mathcal{F}_{(1-\mu)(1-\nu)}(\cdot) \in \mathcal{AC}^1[a,b]$ with the restrictions $-\infty \leq a < t < b \leq \infty, \mu \in (0,1)$ and $\nu \in [0,1]$, then the right-sided Hilfer fractional-order derivative of $h(t)$, symbolically denoted by $({}^{\mathsf{H}}\mathbf{D}^{\mu,\nu}_{a+}h)(t)$, is defined as [24, 25, 26, 27, 30, 63]:

$$({}^{\mathsf{H}}\mathbf{D}^{\mu,\nu}_{a+}h)(t) = \Big({}^{\mathsf{RL}}\mathbf{I}^{\nu(1-\mu)}_{a+} \frac{d}{dt} {}^{\mathsf{RL}}\mathbf{I}^{(1-\nu)(1-\mu)}_{a+} h \Big)(t). \tag{8}$$

For $\nu = 0$, the derivative ${}^{\mathsf{H}}\mathbf{D}^{\mu,\nu}_{a+}$ reduces into the classical Riemann-Liouville fractional-order derivative ${}^{\mathsf{RL}}\mathbf{D}^{\mu}_{a+}$. Also on taking $\nu = 1$ the derivative ${}^{\mathsf{H}}\mathbf{D}^{\mu,\nu}_{a+}$ becomes Caputo fractional-order derivative [33].

1.4. Prabhakar integral

If $h \in \mathcal{L}^1(a,b), 0 \leq a < t \leq b \leq \infty$, then the right-sided Prabhakar integral ${}^{\mathsf{P}}\mathbf{E}^{\gamma}_{\rho,\mu,\omega,a^+}$ of the function $h(t)$ is given as [29, 49, 50]:

$$({}^{\mathsf{P}}\mathbf{E}^{\gamma}_{\rho,\mu,\omega,a^+}h)(t) = h * e^{\gamma}_{\rho,\mu,\omega}(t) = \int_a^t (t-u)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-u)^{\rho}) h(u) du, \tag{9}$$

with $\gamma, \rho, \mu, \omega \in \mathbb{C}$ and $\mathfrak{Re}(\rho) > 0, \mathfrak{Re}(\mu) > 0$. The symbol $e^{\gamma}_{\rho,\mu,\omega}(t)$ in above Eq. (9) is $t^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega t^{\rho})$, and $E^{\gamma}_{\rho,\mu}(\cdot)$ (where $(\cdot)$ denotes argument of the function) is the generalized Mittag-Leffler function, for more details, see [50]. If we take $\gamma = 0$, the integral operator ${}^{\mathsf{P}}\mathbf{E}^{\gamma}_{\rho,\mu,\omega,a^+}$ reduces into the Riemann-Liouville fractional-order integral operator (see Eq. (1)).

1.5. Prabhakar derivative

The Prabhakar derivative is defined as the inverse operator of Prabhakar integral. It is a generalization of the classical Riemann-Liouville derivative. If $h \in \mathcal{L}^1(a,b)$, $0 \le a < t \le b \le \infty$, and $h * e^{-\gamma}_{\rho,m-\mu,\omega}(\cdot) \in \mathcal{W}^{m,1}(a,b), m = \lceil \mu \rceil$, the right-sided Prabhakar derivative ${}^{\mathsf{P}}\mathbf{D}^{\gamma}_{\rho,\mu,\omega,a^+}$ of a function $h(t)$ is given as [29, 49, 50]:

$$({}^{\mathsf{P}}\mathbf{D}^{\gamma}_{\rho,\mu,\omega,a^+}h)(t) = \Big(\frac{d^m}{dt^m}({}^{\mathsf{P}}\mathbf{E}^{-\gamma}_{\rho,m-\mu,\omega,a^+}h)\Big)(t), \tag{10}$$

where $\gamma, \rho, \mu, \omega \in \mathbb{C}$ and $\mathfrak{Re}(\rho) > 0, \mathfrak{Re}(\mu) > 0$.

1.6. Regularized Prabhakar derivative

The regularized Prabhakar derivative operator can be considered as a generalization of Caputo fractional derivative operator. If $h(t) \in \mathcal{AC}^m(a,b), 0 \le a < t \le b \le \infty$, the regularized Prabhakar derivative is defined as [49]:

$$({}^{\mathsf{C}}\mathbf{D}^{\gamma}_{\rho,\mu,\omega,a^+}h)(t) = \Big({}^{\mathsf{P}}\mathbf{E}^{-\gamma}_{\rho,m-\mu,\omega,a^+}\frac{d^m}{dt^m}h\Big)(t). \tag{11}$$

On substituting $\gamma = 0$ the derivative $({}^{\mathsf{C}}\mathbf{D}^{\gamma}_{\rho,\mu,\omega,a^+}h)(t)$ reduces into Caputo derivative ${}^{\mathsf{C}}\mathbf{D}^{\mu}_{a+}h(t)$, defined by Eq. (7) in the subsection 1.2.

1.7. Hilfer-Prabhakar derivative

The Hilfer-Prabhakar derivatives (also known as the generalized Hilfer derivative) is emerging as a useful differential operator, particularly in mathematical physics and other branches of applied mathematics. Garra et al. [19] have described the dynamics of the generalized renewal stochastic process and some other classical equations of mathematical physics in terms of Hilfer-Prabhakar derivatives.

If $h \in \mathcal{L}^1(a,b), h * e^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega}(\cdot) \in \mathcal{AC}^1(a,b)$ with the restrictions $0 \le a < t \le b \le \infty$, $\mu \in (0,1)$, and $\nu \in [0,1]$, then the Hilfer-Prabhakar derivative is defined as [19]:

$$({}^{\mathsf{HP}}\mathbf{D}^{\gamma,\mu,\nu}_{\rho,\omega,a^+}h)(t) = \Big({}^{\mathsf{P}}\mathbf{E}^{-\gamma\nu}_{\rho,\nu(1-\mu),\omega,a^+}\frac{d}{dt}({}^{\mathsf{P}}\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\mu)(1-\nu),\omega,a^+}h)\Big)(t), \tag{12}$$

where $\omega, \rho, \gamma \in \mathbb{C}$ with $\mathfrak{Re}(\rho) > 0$. Particularly, If we put $\gamma = 0$, the Hilfer-Prabhakar derivative becomes the Hilfer derivative given in above Eq. (8). The remarkable property of the Hilfer-Prabhakar derivative is that it interpolates between the Prabhakar derivative and its regularized version, given in Eq. (10) and Eq. (11), respectively.

1.8. The Lorenzo-Hartley's function

Special functions arise ubiquitously in solutions of fractional differential equations. The Agarwal's function, Mittag-Leffler functions (with one, two & three parameters), Erdélyi's function, Robotnov & Hartley's function, Miller & Ross's function are some of the appear naturally in the solution of various differential equations of integer and non-integer orders. Lorenzo and Hartley [34] investigated a multivalued generalization of standard exponential function known as Lorenzo-Hartley's G-function (LHGF), denoted as $G_{\{\rho,\beta,\delta\}}(\omega, v, t)$. Being an eigenfunction, all the order fractional differ-integrals of LHGF appear in terms of LHGF (with suitably modified parameters). In a recent monograph [35], it is shown that such generalized functions have a great potential in investigations of scientific applications pertaining to Galactic classification, Shell morphology, Weather prediction, etcetera. The infinite series representation of LHGF given as:

$$G_{\{\rho,\beta,\delta\}}(\omega, v, t) = \sum_{k=0}^{\infty} \frac{(\delta)_k \omega^k (t-v)^{(k+\delta)\rho-\beta-1}}{k!\Gamma((k+\delta)\rho-\beta)}, \text{ with } \mathfrak{Re}(\rho\delta-\beta) > 0, \tag{13}$$

where $(\delta)_k$ is the generalization of factorial (also known as rising factorial or Pochhammer's symbol), is defined as:

$$(\delta)_0 = 1, (\delta)_1 = \delta, (\delta)_2 = (\delta)(\delta+1), ..., (\delta)_n = (\delta)(\delta+1)...(\delta+n-1). \tag{14}$$

On substituting $v = 0$ Eq. (13) reduces in to following convenient form:

$$G_{\{\rho,\beta,\delta\}}(\omega, 0, t) = G_{\{\rho,\beta,\delta\}}(\omega, t) = \sum_{k=0}^{\infty} \frac{(\delta)_k \omega^k t^{(k+\delta)\rho-\beta-1}}{k!\Gamma((k+\delta)\rho-\beta)}, \text{ with } \mathfrak{Re}(\rho\delta-\beta) > 0. \tag{15}$$

A number of functions have direct and elegant relationships with the LHGF $G_{\{\rho,\beta,\delta\}}(\omega, b, t)$, for more details one can refer recent investigation [46].

A fractional function LHGF is gaining importance in applications and analysis as it can handle increased time-domain complexity. In [64] Yang has discussed generalized fractional derivatives and integrals involving LHGFs (of one and two parameters) in the kernel and illustrates some applications in applied sciences. In a most recent monograph, Yang et al. [65] have demonstrated applications of such fractional operators for the investigations of models pertaining to viscoelasticity. Chaurasia and Pandey [11, 12] have extended the work of Haubold and Mathai [23] and studied computable closed-forms of some generalized fractional kinetic equations in terms of LHGF. Saxena et al. [57] have used LHGF in the investigation of generalized fractional kinetic equations. Goufo [17] have applied this function in the study related to bio-mathematical analysis associated with cellulose degradation dynamics. For some more details about LHGF one can also refer to Mahmood et al. [36], Saha et al. [53], Shakeel et al. [58] and recent investigations by Pandey [45, 46] .

2. A unification of fractional Volterra-type integro-differential equations

In this section, after presenting a chronological review pertaining to the development of a Volterra-type fractional integro-differential equation (FIDE), we propose a unified family of Volterra-type fractional integro-differential equations involving LHGF in the kernel and a generalized Hilfer derivative. It is emphasized that the solution obtained is also represented in a closed-form of LHGF. For the sake of simplified computations via LHGF we have assumed that the order of fractional derivative lies between 0 and 1. Moreover, we discuss the convergence of the solution by the method recently used by Giusti and Colombaro in [20] during the investigation of a generalized Viscoelastic model.

Dattoli et al. [14] studied the following first-order integro-differential equation of Volterra-type involving exponential function in the kernel:

$$\frac{d}{dt}(h(t)) = -i\pi g_0 \int_0^t \zeta h(t-\zeta)e^{i\omega\zeta}d\zeta, \ \ 0 \leq t \leq 1, \ \text{ with } \ h(0) = h_0 \text{ and } g_0, \omega \in \mathbb{R}, \tag{16}$$

and discussed analytical treatment that describes the unsaturated behaviour of the free-electron laser equation (for other details, see [13]).

In this direction, Boyadjiev et al. [9] proposed following fractional analogue form of the Volterra-type integro-differential Eq. (16) and examined analytic and numerical behaviour of the solution:

$$\left({}^{\mathsf{RL}}\mathbf{D}_t^{\mu} h\right)(t) = \ell \int_0^t \zeta h(t-\zeta)e^{i\omega\zeta}d\zeta, \quad 0 \leq t \leq 1, \tag{17}$$

with ${}^{\mathsf{RL}}\mathbf{D}_t^{\mu-j} h(t)|_{t=0} = h_j \in \mathbb{R}$ $(j = 1, 2, 3, \ldots, n)$, and where $\mu, \ell \in \mathbb{C}$, $(n-1) < \Re e(\mu) \leq n$, $n = -[-\Re e(\mu)]$, $\omega \in \mathbb{R}$. The symbol ${}^{\mathsf{RL}}\mathbf{D}_t^{\mu}$ in above Eq. (17) denotes is the well-known Riemann-Liouville fractional derivative of order μ.
On substituting $\vartheta = (t-\zeta)$, the Eq. (17) reduces into the following alternative form:

$$\left({}^{\mathsf{RL}}\mathbf{D}_t^{\mu} h\right)(t) = \ell \int_0^t (t-\vartheta)h(\vartheta)e^{i\omega(t-\vartheta)}d\vartheta. \tag{18}$$

Concurrently, Boyadjiev et al. [10] studied and investigated non-homogeneous FIDE of the form:

$$\left({}^{\mathsf{RL}}\mathbf{D}_t^{\mu} h\right)(t) = \ell \int_0^t \zeta h(t-\zeta)e^{i\omega\zeta}d\zeta + \beta' e^{i\omega t}, \quad 0 \leq t \leq 1, \tag{19}$$

with ${}^{\mathsf{RL}}\mathbf{D}_t^{\mu-j} h(t)|_{t=0} = h_j \in \mathbb{R}$ $(j = 1, 2, 3, \ldots n)$, where $\mu, \beta', \ell \in \mathbb{C}; \omega \in \mathbb{R}; (n-1) < \Re e(\mu) \leq n,$ and $n = -[-\Re e(\mu)]$. An alternative form of the above FIDE can also be obtained as:

$$\left({}^{\mathsf{RL}}\mathbf{D}_t^{\mu} h\right)(t) = \ell \int_0^t (t-\vartheta) h(\vartheta) e^{i\omega(t-\vartheta)} d\vartheta + \beta' e^{i\omega t}. \tag{20}$$

Al-Shammery et al. [4] discussed following generalized form of FIDE and extended the idea of Boyadjiev et al. [10]:

$$\left({}^{\mathsf{RL}}\mathbf{D}_t^{\mu} h\right)(t) = \ell \int_0^t \zeta^{\kappa} h(t-\zeta) e^{i\omega\zeta} d\zeta + \beta' e^{i\omega t}, \quad 0 \le t \le 1, \tag{21}$$

with $\mu, \beta', \ell \in \mathbb{C}$, and $\omega \in \mathbb{R}$, and $\kappa > -1$. The above FIDE can be alternatively put in following form:

$$\left({}^{\mathsf{RL}}\mathbf{D}_t^{\mu} h\right)(t) = \ell \int_0^t (t-\vartheta)^{\kappa} h(\vartheta) e^{i\omega(t-\vartheta)} d\vartheta + \beta' e^{i\omega t}, \quad 0 \le t \le 1, \tag{22}$$

with $\mu, \beta', \ell \in \mathbb{C}, \omega \in \mathbb{R}$ and $\kappa > -1$.

In continuation Saxena and Kalla [55] considered following extension of FIDE involving Kummer's hypergeometric function [38, 39]:

$$\left({}^{\mathsf{RL}}\mathbf{D}_t^{\mu} h\right)(t) = \ell \int_0^t \zeta^{\kappa} h(t-\zeta) \Phi(b, \kappa+1; i\omega\zeta) d\zeta + \rho' t^{\gamma} \Phi(\beta', \gamma+1; i\omega t), \ 0 \le t \le 1, \tag{23}$$

with $\mu, b, \beta', \rho', \ell \in \mathbb{C}, \omega \in \mathbb{R}$, and $\kappa > -1$. The above FIDE (23) alternatively can be put in following form:

$$\left({}^{\mathsf{RL}}\mathbf{D}_t^{\mu} h\right)(t) = \ell \int_0^t (t-\vartheta)^{\kappa} h(\vartheta) \Phi(b, \kappa+1; i\omega(t-\vartheta)) d\vartheta + \rho' t^{\gamma} \Phi(\beta', \gamma+1; i\omega t), \tag{24}$$

$0 \le t \le 1$, with $\mu, b, \beta', \rho', \ell \in \mathbb{C}$ and $\omega \in \mathbb{R}$ and $\kappa > -1$.

At the same time Kilbas et al. [28] have proposed and studied following interesting and generalized form of the of above Eq. (24) which involves the well-known Mittag-Leffler function [22] in the kernel and a general function $\psi(t)$:

$$\left({}^{\mathsf{RL}}\mathbf{D}_{a+}^{\mu} h\right)(t) = \ell \int_a^t (t-\vartheta)^{(\kappa-1)} E_{\rho,\kappa}^{\delta}(\omega(t-\vartheta)^{\rho}) h(\vartheta) d\vartheta + \psi(t), \tag{25}$$

where $\mu, \rho, \kappa, \delta$ and $\omega \in \mathbb{C}$ (with $\mathfrak{Re}(\kappa) > 0$, $\mathfrak{Re}(\mu) > 0$, $\mathfrak{Re}(\rho) > 0$).

Now, we propose a unified family of fractional integro-differential equations of Volterra-type. Such unifications may deduce several interesting forms of well-known (maybe also new) fractional integro-differential equations and provide a common framework for computation of numerous problems in applied sciences.

Theorem 2.1. *If $\psi(t)$ is a general function with $h(t) \in \mathcal{L}^1(0,\infty); \gamma,\delta,\rho,\omega,\alpha,\beta \in \mathbb{C}$; $0<\mu<1, 0\le\nu\le 1$; $\mathfrak{Re}(\gamma)\ge 0$, $\mathfrak{Re}(\delta)\ge 0$, $\mathfrak{Re}(\rho\delta-\beta)>0$, $\mathfrak{Re}(\rho)>0$, $\mathfrak{Re}(\omega)>0$, then for FIDE:*

$$\left({}^{\mathsf{HP}}\boldsymbol{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t)+\alpha\int_0^t G_{\{\rho,\beta,\delta\}}(\omega,x,t)h(x)dx=\psi(t), \tag{26}$$

with $\left({}^{\mathsf{P}}\boldsymbol{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0+}h(t)\right)_{t=0+}=c$, *the following solution holds:*

$$h(t)=\psi(t)*\sum_{k=0}^{\infty}(-\alpha)^k\left[G_{\{\rho,[(\beta-\mu-\rho\delta)k-\mu+\rho\{(\gamma+\delta)k+\gamma\}],[(\gamma+\delta)k+\gamma]\}}(\omega,0,t)\right]$$

$$+c\sum_{k=0}^{\infty}(-\alpha)^k\left[G_{\{\rho,[(\beta-\mu-\rho\delta)k-\mu-\nu(1-\mu)+\rho\{(\gamma+\delta)k+\gamma-\nu\gamma\}],[(\gamma+\delta)k+\gamma-\nu\gamma]\}}(\omega,0,t)\right], \tag{27}$$

provided the sums on the RHS of above Eq. (27) converges.

Proof. The proof of the theorem is based on the Laplace transform method [47]. Applying the Laplace transform both the sides of the above integro-differential Eq. (26), and using the following well-known result pertaining to the Laplace transform of Hilfer-Prabhakar derivative operator (see for more details [19, 46, 49]):

$$L\left({}^{\mathsf{HP}}\mathbf{D}^{\gamma,\mu,\nu}_{\rho,\omega,a^+}h\right)(s)=L\left(\left({}^{\mathsf{P}}\mathbf{E}^{-\gamma\nu}_{\rho,\nu(1-\mu),\omega,a^+}\frac{d}{dt}\left({}^{\mathsf{P}}\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\mu)(1-\nu),\omega,a^+}h\right)\right)(s)$$

$$=s^{\mu}[1-\omega s^{-\rho}]^{\gamma}L[h](s)-s^{-\nu(1-\mu)}[1-\omega s^{-\rho}]^{\gamma\nu}\left({}^{\mathsf{P}}\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\mu)(1-\nu),\omega,a^+}h(t)\right)_{t=a+}, \tag{28}$$

we get

$$s^{\mu}(1-\omega s^{-\rho})^{\gamma}\bar{h}(s)-cs^{-\nu(1-\mu)}(1-\omega s^{-\rho})^{\gamma\nu}+\alpha\frac{s^{\beta-\rho\delta}}{(1-\omega s^{-\rho})^{\delta}}\bar{h}(s)=\bar{g}(s), \tag{29}$$

where $\bar{h}(s)$ and $\bar{g}(s)$ are the Laplace transforms of $h(t)$ and $\psi(t)$, respectively. Also, Eq. (29) can be rewritten as:

$$\bar{h}(s)\left[s^{\mu}(1-\omega s^{-\rho})^{\gamma}+\alpha\frac{s^{\beta-\rho\delta}}{(1-\omega s^{-\rho})^{\delta}}\right]=\bar{g}(s)+cs^{-\nu(1-\mu)}(1-\omega s^{-\rho})^{\nu\gamma}, \tag{30}$$

or alternatively

$$\bar{h}(s)=\frac{\bar{g}(s)}{s^{\mu}(1-\omega s^{-\rho})^{\gamma}}\left[1+\frac{\alpha}{s^{\mu+\rho\delta-\beta}(1-\omega s^{-\rho})^{(\gamma+\delta)}}\right]^{-1}$$

$$+\frac{s^{-\nu(1-\mu)}(1-\omega s^{-\rho})^{\nu\gamma}c}{s^{\mu}(1-\omega s^{-\rho})^{\gamma}}\left[1+\frac{\alpha}{s^{\mu+\rho\delta-\beta}(1-\omega s^{-\rho})^{(\gamma+\delta)}}\right]^{-1}. \tag{31}$$

By the use of binomial series expansion the above Eq. (31) reduces into following computable series:

$$\bar{h}(s)=\bar{g}(s)\sum_{k=0}^{\infty}(-\alpha)^k\frac{s^{(\beta-\mu-\rho\delta)k-\mu}}{(1-\omega s^{-\rho})^{(\gamma+\delta)k+\gamma}}+c\sum_{k=0}^{\infty}(-\alpha)^k\frac{s^{(\beta-\mu-\rho\delta)k-\mu-\nu(1-\mu)}}{(1-\omega s^{-\rho})^{(\gamma+\delta)k+\gamma-\nu\gamma}}. \quad (32)$$

It is easy to see that the expressions involved in Eq. (31) will be there in the existence provided both the infinite series are absolutely convergent power series, i.e., we must have following condition:

$$\left|\frac{\alpha}{s^{\mu+\rho\delta-\beta}(1-\omega s^{-\rho})^{(\gamma+\delta)}}\right|<1. \quad (33)$$

By the application of the well-known convolution theorem of the Laplace transform and taking inverse Laplace transform on both the sides of above Eq. (32) , we arrive on the desired solution of Eq. (26), given in Eq. (27).

Computation of the solution obtained in Eq. (27) is less trivial and based on the convergence of each term involved therein. The first expression involves convolution of the function $\psi(t)$ with each term of infinite series of LHGF, i.e.

$$\psi(t)*\sum_{k=0}^{\infty}(-\alpha)^k\Big[G_{\{\rho,[(\beta-\mu-\rho\delta)k-\mu+\rho\{(\gamma+\delta)k+\gamma\}],[(\gamma+\delta)k+\gamma]\}}(\omega,0,t)\Big]. \quad (34)$$

The convergence of the above expression, Eq. (34), is based on the convergence of following infinite series consisting repeated series of LHGF:

$$\sum_{k=0}^{\infty}(-\alpha)^k\Big[G_{\{\rho,[(\beta-\mu-\rho\delta)k-\mu+\rho\{(\gamma+\delta)k+\gamma\}],[(\gamma+\delta)k+\gamma]\}}(\omega,0,t)\Big],$$

which can be rewritten by the use of series form of LHGF as:

$$\sum_{k=0}^{\infty}(-\alpha)^k\sum_{n=0}^{\infty}\frac{((\gamma+\delta)k+\gamma)_n\omega^n t^{\{(n+(\gamma+\delta)k+\gamma)\rho-(\beta-\mu-\rho\delta)k+\mu-\rho((\gamma+\delta)k+\gamma)-1\}}}{n!\Gamma\{(n+(\gamma+\delta)k+\gamma)\rho-(\beta-\mu-\rho\delta)k+\mu-\rho((\gamma+\delta)k+\gamma)\}}, \quad (35)$$

or equivalently

$$\sum_{k=0}^{\infty}(-\alpha)^k\sum_{n=0}^{\infty}\frac{((\gamma+\delta)k+\gamma)_n\omega^n t^{\{\rho n+(\rho\delta+\mu-\beta)k+\mu-1\}}}{n!\Gamma\{\rho n+(\rho\delta+\mu-\beta)k+\mu\}}. \quad (36)$$

In order to prove the absolute convergence of the series labelled by k, we need to show that the series

$$\sum_{k=0}^{\infty}(-\alpha)^k\frac{((\gamma+\delta)k+\gamma)_n\omega^n t^{\{\rho n+(\rho\delta+\mu-\beta)k+\mu-1\}}}{n!\Gamma\{\rho n+(\rho\delta+\mu-\beta)k+\mu\}}, \quad (37)$$

is absolutely convergent for each (fixed) $n \in \mathbb{N} \cup \{0\}$ (for more details, see [5]).
Let us define

$$u_k(n,t) = (-\alpha)^k \frac{((\gamma+\delta)k+\gamma)_n \omega^n t^{\{\rho n+(\rho\delta+\mu-\beta)k+\mu-1\}}}{n!\Gamma\{\rho n+(\rho\delta+\mu-\beta)k+\mu\}}$$

$$= (-\alpha)^k \frac{\Gamma\{(\gamma+\delta)k+\gamma+n\}}{\Gamma\{(\gamma+\delta)k+\gamma\}} \frac{(\omega t^\rho)^n t^{\{(\rho\delta+\mu-\beta)k+\mu-1\}}}{n!\Gamma\{\rho n+(\rho\delta+\mu-\beta)k+\mu\}}, \tag{38}$$

using the asymptotic behaviour of the ratio of gamma functions we get

$$\left|\frac{u_{k+1}(n,t)}{u_k(n,t)}\right| \sim \left|\frac{(-\alpha)t^{(\rho\delta+\mu-\beta)}}{k(\rho\delta+\mu-\beta)}\right| \quad \forall\ t>0,\ \forall\ n \in \mathbb{N}\cup\{0\}, \tag{39}$$

hence for $k \to \infty$ we get

$$\lim_{k\to\infty}\left|\frac{u_{k+1}(n,t)}{u_k(n,t)}\right| = 0,\ \forall\ t>0, \forall\ n \in \mathbb{N}\cup\{0\}, \tag{40}$$

which indicates that the absolute convergence of the series involved in the first term of Eq. (27). Also, if the function $\psi(t)$ is continuous and suitably defined in $\mathcal{L}^1(0,\infty)$ then the convolution must be convergent and the first term of Eq. (27) must be convergent. The convergence of the second term of Eq. (27) can also be investigated in a similar manner, thus we omit the details here. □

3. Certain Volterra-type fractional integro-differential equations based on the family of unified fractional Volterra-type integro-differential equations

The above-discussed family of Volterra-type FIDE is general in nature and unifies several elegant and interesting results proposed by eminent scholars. In the present section, based on Theorem 2.1 we deduce some of the corollaries which may be directly applicable in different fields of sciences, such as laser, nuclear, astrophysics, thermal analysis, heat transfer etcetera.
For $\delta = 0; \beta = -\eta$, Theorem 2.1 reduces into the following result recently investigated by Pandey [46]:

Corollary 3.1. *If* $h(t) \in \mathcal{L}^1(0,\infty)$; $\gamma,\ \rho,\ \omega,\ \alpha,\ \eta \in \mathbb{C}$; $0 < \mu < 1, 0 \le \nu \le 1; \mathfrak{Re}(\gamma) \ge 0, \mathfrak{Re}(\delta) \ge 0,\ \mathfrak{Re}(\eta) > 0, \mathfrak{Re}(\rho) > 0, \mathfrak{Re}(\omega) > 0$, *then for the FIDE:*

$$\left({}^{\mathsf{HP}}\boldsymbol{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t) + \alpha\int_0^t G_{\{\rho,-\eta,0\}}(\omega,x,t)h(x)dx = \psi(t), \tag{41}$$

or, equivalently

$$\left({}^{\mathsf{HP}}\boldsymbol{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t) + \frac{\alpha}{\Gamma(\eta)}\int_0^t (t-x)^{\eta-1}h(x)dx = \psi(t), \tag{42}$$

with $\left({}^{\mathsf{P}}\boldsymbol{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0+}h(t)\right)_{t=0+} = c$, *following solution holds:*

$$h(t) = \psi(t) * \sum_{k=0}^{\infty}(-\alpha)^k \Big[G_{\{\rho,[\rho\gamma(k+1)-(\eta+\mu)k-\mu],[\gamma(k+1)]\}}(\omega,0,t)\Big]$$

$$+c\sum_{k=0}^{\infty}(-\alpha)^k \Big[G_{\{\rho,[\nu(\mu-1)-\mu-(\eta+\mu)k+\rho\{\gamma(k+1)-\nu\gamma\}],[\gamma(k+1)-\nu\gamma]\}}(\omega,0,t)\Big], \tag{43}$$

provided the sums on the RHS of above Eq. (43) converges.

Remark. The results presented as Corollaries 7, 8 and 9 in [46] can also be deduced as the particular cases of the theorem 2.1. For more details, see [2] and [63].

On taking $\gamma = 0$ in Theorem 2.1, we arrive on the following corollary pertaining to certain family of Volterra-type FIDE based on the Hilfer derivative that involves LHGF in the kernel.

Corollary 3.2. *If* $h(t) \in \mathcal{L}^1(0,\infty)$; $0 < \mu < 1, 0 \le \nu \le 1$; $\alpha,\rho,\beta,\delta \in \mathbb{C}$; $\mathfrak{Re}(\rho\delta-\beta) > 0$, $\mathfrak{Re}(\omega) > 0$ *then for FIDE of the form:*

$$\left({}^{\mathsf{H}}\mathcal{D}^{\mu,\nu}_{0+}h\right)(t) + \alpha\int_0^t G_{\{\rho,\beta,\delta\}}(\omega,x,t)h(x)dx = \psi(t), \tag{44}$$

with $\left({}^{\mathsf{RL}}I^{(1-\mu)(1-\nu)}_{0+}h(t)\right)_{t=0+} = c$, *following solution holds:*

$$h(t) = \psi(t) * \sum_{k=0}^{\infty}(-\alpha)^k \Big[G_{\{\rho,[(\beta-\mu)k-\mu],\delta k\}}(\omega,0,t)\Big]$$

$$+c\sum_{k=0}^{\infty}(-\alpha)^k \Big[G_{\{\rho,[(\beta-\mu)k-\mu-\nu(1-\mu)],[\delta k]\}}(\omega,0,t)\Big], \tag{45}$$

provided the sums on the RHS of above Eq. (45) converges.

On setting $\nu = 0$ in Corollary 3.2, we get the following form of Volterra-type FIDE:

Corollary 3.3. *If* $h(t) \in \mathcal{L}^1(0,\infty)$; $0 < \mu < 1$; $\alpha,\rho,\beta,\delta \in \mathbb{C}$; $\mathfrak{Re}(\rho\delta-\beta) > 0$, $\mathfrak{Re}(\omega) > 0$, *then for FIDE:*

$$\left({}^{\mathsf{RL}}\boldsymbol{D}^{\mu}_{0+}h\right)(t) + \alpha\int_0^t G_{\{\rho,\beta,\delta\}}(\omega,x,t)h(x)dx = \psi(t), \tag{46}$$

with $\left({}^{\mathsf{RL}}\boldsymbol{I}^{(1-\mu)}_{0+}h(t)\right)_{t=0+} = c$, *following solution holds:*

$$h(t) = \psi(t) * \sum_{k=0}^{\infty}(-\alpha)^k \Big[G_{\{\rho,[(\beta-\mu)k-\mu],\delta k\}}(\omega,0,t)\Big]$$

$$+c\sum_{k=0}^{\infty}(-\alpha)^{k}\Big[G_{\{\rho,[(\beta-\mu)k-\mu],\delta k\}}(\omega,0,t)\Big], \tag{47}$$

provided the sums on the RHS of above Eq. (47) converges.

Remark. If we substitute $\alpha = -\ell(\Gamma(\kappa+1))$ with $\rho \to 1$; $\beta \to (\varpi - \kappa + 1)$; $\delta \to \varpi$; $\omega \to i\omega$ and $\psi(t) = \rho'\Gamma(\gamma+1)G_{\{1,(\beta'-\gamma+1),\beta'\}}(i\omega,0,t)$ in the above Eq. (46) of Corollary 3.3, we arrive on the following Volterra-type FIDE:

$$\left({}^{\mathsf{RL}}\mathbf{D}^{\mu}_{0+}h\right)(t) = \ell(\Gamma(\kappa+1))\int_0^t G_{\{1,(\varpi-\kappa+1),\varpi\}}(i\omega,x,t)h(x)dx$$
$$+\rho'\Gamma(\gamma+1)G_{\{1,(\beta'-\gamma+1),\beta'\}}(i\omega,0,t), \tag{48}$$

with $\left({}^{\mathsf{RL}}\mathbf{I}^{(1-\mu)}_{0+}h(t)\right)_{t=0+} = c$. The FIDE in Eq. (48) is equivalent to the result studied by Saxena and Kalla [55], discussed in Eq. (24).

Remark. Using the relation given in [46], Eq. (25), in above Eq. (46), we arrive on following FIDE:

$$\left({}^{\mathsf{RL}}\mathbf{D}^{\mu}_{0+}h\right)(t) + \alpha\int_0^t (t-x)^{\rho\delta-\beta-1}E^{\delta}{}_{\rho,(\rho\delta-\beta)}(\omega(t-x)^{\rho})(h(x)dx = \psi(t), \tag{49}$$

with $\left({}^{\mathsf{RL}}\mathbf{I}^{(1-\mu)}_{0+}h(t)\right)_{t=0+} = c$, which on substituting $\alpha = -\ell$; $\rho\delta - \beta = \kappa$ yields the well-known Volterra-type integro-differential equation investigated by Kilbas et al. [28], given in above Eq. (25).

On taking $\nu = 1$ in Corollary 3.2, we get following family of Volterra-type FIDE in terms of the Caputo derivative involving LHGF in the kernel.

Corollary 3.4. *If* $h(t) \in \mathcal{L}^1(0,\infty)$; $0 < \mu < 1$; $\alpha,\rho,\beta,\delta \in \mathbb{C}$; $\mathfrak{Re}(\rho\delta-\beta) > 0$, $\mathfrak{Re}(\omega) > 0$, *then for FIDE:*

$$\left({}^{\mathsf{C}}\boldsymbol{D}^{\mu}_{0+}h\right)(t) + \alpha\int_0^t G_{\{\rho,\beta,\delta\}}(\omega,x,t)h(x)dx = \psi(t), \tag{50}$$

with $h(t)_{t=0+} = c$, *following solution holds:*

$$h(t) = \psi(t) * \sum_{k=0}^{\infty}(-\alpha)^{k}\Big[G_{\{\rho,[(\beta-\mu)k-\mu],\delta k\}}(\omega,0,t)\Big]$$
$$+c\sum_{k=0}^{\infty}(-\alpha)^{k}\Big[G_{\{\rho,[(\beta-\mu)k-\mu-(1-\mu)],\delta k\}}(\omega,0,t)\Big], \tag{51}$$

provided the sum on the RHS of above Eq. (51) converges.

Remark. The detailed analysis of the corollaries concerning convergence discussed in this section can be done exactly in the same way as we have proposed in Theorem 2.1.

4. Certain examples pertaining to the unified family of Volterra-type fractional integro-differential equations

In this section, we investigate some applications of Theorem 2.1 by considering some particular forms of the function $\psi(t)$. Let's consider the case when $\psi(t) = G_{\{\rho,\eta,\xi\}}(\omega, 0, t)$, we arrive on the following result:

Example 4.1. If $h(t)\in\mathcal{L}^1(0,\infty)$; $0<\mu<1$, $0\le\nu\le1$; $\gamma\ge0$; $\rho,\omega,\alpha,\eta,\beta,\delta,\xi,\lambda\in\mathbb{C}$; $\mathfrak{Re}(\rho\delta-\beta)>0$, $\mathfrak{Re}(\rho\xi-\eta)>0$, $\mathfrak{Re}(\rho)>0$, $\mathfrak{Re}(\omega)>0$, then for FIDE:

$$\left({}^{\mathsf{HP}}\mathbf{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t)+\alpha\int_0^t G_{\{\rho,\beta,\delta\}}(\omega,x,t)h(x)dx=\psi(t)=\lambda G_{\{\rho,\eta,\xi\}}(\omega,0,t), \tag{52}$$

with $\left({}^{\mathsf{P}}\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0+}h(t)\right)_{t=0+}=c,$ following solution holds:

$$h(t)=c\sum_{k=0}^{\infty}(-\alpha)^k\Big[G_{\{\rho,[(\beta-\mu+\rho\gamma)k-\nu(1-\mu)-\mu+\rho\gamma(1-\nu)],[(\gamma+\delta)k+\gamma(1-\nu)]\}}(\omega,0,t)\Big]$$

$$+\lambda\sum_{k=0}^{\infty}(-\alpha)^k\Big[G_{\{\rho,[(\beta-\mu+\rho\gamma)k+(\eta-\mu+\rho\gamma)],[(\gamma+\delta)k+\gamma+\xi]\}}(\omega,0,t)\Big], \tag{53}$$

provided the sum on RHS of above Eq. (53) converges.

Particularly, if we substitute $\lambda=0$ in the above example we arrive on the following homogeneous FIDE:

$$\left({}^{\mathsf{HP}}\mathbf{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t)+\alpha\int_0^t G_{\{\rho,\beta,\delta\}}(\omega,x,t)h(x)dx=0, \tag{54}$$

with $\left({}^{\mathsf{P}}\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0+}h(t)\right)_{t=0+}=c$, then following solution holds:

$$h(t)=c\sum_{k=0}^{\infty}(-\alpha)^k\Big[G_{\{\rho,[(\beta-\mu+\rho\gamma)k-\nu(1-\mu)-\mu+\rho\gamma(1-\nu)],[(\gamma+\delta)k+\gamma(1-\nu)]\}}(\omega,0,t)\Big], \tag{55}$$

provided the sum on RHS of above Eq. (55) converges.

Furthermore, if we substitute $\delta=0$; $\beta=-\tau$ with the conditions $\mathfrak{Re}(\rho)>0$, $\mathfrak{Re}(\tau)>0, \mathfrak{Re}(\rho\xi-\eta)>0$, $\mathfrak{Re}(\omega)>0$ in Example 4.1, we arrive on following FIDE:

$$\left({}^{\mathsf{HP}}\mathbf{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t)+\alpha\int_0^t G_{\{\rho,-\tau,0\}}(\omega,x,t)h(x)dx=\lambda G_{\{\rho,\eta,\xi\}}(\omega,0,t), \tag{56}$$

or equivalently (applying the relation [46], Eq. (16))

$$\left({}^{\mathsf{HP}}\mathbf{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t)+\frac{\alpha}{\Gamma(\tau)}\int_0^t(t-x)^{\tau-1}h(x)dx=\lambda G_{\{\rho,\eta,\xi\}}(\omega,0,t), \tag{57}$$

with $\left({}^{\mathsf{P}}\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0+}h(t)\right)_{t=0+}=c$, which has its solution as:

$$h(t)=c\sum_{k=0}^{\infty}(-\alpha)^k\left[G_{\{\rho,[(\rho\gamma-\tau-\mu)k-\nu(1-\mu)-\mu+\rho\gamma(1-\nu)],[\gamma k+\gamma(1-\nu)]\}}(\omega,0,t)\right]$$
$$+\lambda\sum_{k=0}^{\infty}(-\alpha)^k\left[G_{\{\rho,[(\rho\gamma-\tau-\mu)k+(\eta-\mu+\rho\gamma)],[\gamma k+\gamma+\xi]\}}(\omega,0,t)\right], \tag{58}$$

provided the sum on the RHS of above Eq. (58) converges.

Let us consider the case when the function $\psi(t)=\lambda G_{\{\rho,-\eta,0\}}(\omega,0,t)$, then by the Theorem 2.1 we can deduce following particular example:

Example 4.2. If $h(t)\in\mathcal{L}^1(0,\infty)$; $0<\mu<1$, $0\le\nu\le1$; $\gamma\ge0$; $\rho,\omega,\alpha,\eta,\beta,\delta\in\mathbb{C}$; $\mathfrak{Re}(\rho\delta-\beta)>0$, $\mathfrak{Re}(\eta)>0$, $\mathfrak{Re}(\rho)>0$, $\mathfrak{Re}(\omega)>0$ then for FIED:

$$\left({}^{\mathsf{HP}}\mathbf{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t)+\alpha\int_0^t G_{\{\rho,\beta,\delta\}}(\omega,x,t)h(x)dx=\lambda G_{\{\rho,-\eta,0\}}(\omega,0,t), \tag{59}$$

or equivalently (applying the relation [46], Eq. (21))

$$\left({}^{\mathsf{HP}}\mathbf{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t)+\alpha\int_0^t G_{\{\rho,\beta,\delta\}}(\omega,x,t)h(x)dx=\lambda\frac{t^{\eta-1}}{\Gamma(\eta)}, \tag{60}$$

with $\left({}^{\mathsf{P}}\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0+}h(t)\right)_{t=0+}=c$, following solution holds:

$$h(t)=c\sum_{k=0}^{\infty}(-\alpha)^k\left[G_{\{\rho,[(\beta-\mu+\rho\gamma)k-\nu(1-\mu)-\mu+\rho\gamma(1-\nu)],[(\gamma+\delta)k+\gamma(1-\nu)]\}}(\omega,0,t)\right]$$
$$+\lambda\sum_{k=0}^{\infty}(-\alpha)^k\left[G_{\{\rho,[(\beta-\mu+\rho\gamma)k+(\rho\gamma-\eta-\mu)],[(\gamma+\delta)k+\gamma]\}}(\omega,0,t)\right], \tag{61}$$

provided the sum on RHS of above Eq. (61) converges.

Moreover, for $\delta=0;\beta=-\sigma$ with $\mathfrak{Re}(\sigma)>0,\mathfrak{Re}(\eta)>0$ FIDE, presented as Eq. (59), reduces into following form:

$$\left({}^{\mathsf{HP}}\mathbf{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t)+\alpha\int_0^t G_{\{\rho,-\sigma,0\}}(\omega,x,t)h(x)dx=\lambda G_{\{\rho,-\eta,0\}}(\omega,0,t), \tag{62}$$

which equivalently can be rewritten as

$$\left({}^{\mathsf{HP}}\mathbf{D}^{\gamma,\mu,\nu}_{\rho,\omega,0+}h\right)(t)+\frac{\alpha}{\Gamma(\sigma)}\int_0^t(t-x)^{\sigma-1}h(x)dx=\lambda\frac{t^{\eta-1}}{\Gamma(\eta)},\tag{63}$$

with $\left({}^{\mathsf{P}}\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0+}h\right)_{t=0+}=c$ has its solution as:

$$\begin{aligned}h(t)=c&\sum_{k=0}^{\infty}(-\alpha)^k\left[G_{\{\rho,[(\rho\gamma-\sigma-\mu)k-\nu(1-\mu)-\mu+\rho\gamma(1-\nu)],[\gamma k+\gamma(1-\nu)]\}}(\omega,0,t)\right]\\&+\lambda\sum_{k=0}^{\infty}(-\alpha)^k\left[G_{\{\rho,[(\rho\gamma-\sigma-\mu)k+(\rho\gamma-\eta-\mu)],[\gamma k+\gamma]\}}(\omega,0,t)\right],\end{aligned}\tag{64}$$

provided the sum on RHS of above Eq. (64) converges.

The solutions of Theorem 2.1, its corollaries, and associated examples are obtained in terms of LHGF where we tactically assumed that the range of different parameters are taken in such a way that the obtained solutions must be convergent.

5. Applications in Free-electron laser (FEL) equations

To demonstrate applications of the presented unified family of fractional Volterra-type integro-differential equation, we deduce two fractional-order generalizations of free-electron laser equations involving LHGF in the kernel as the special cases of Example 4.1.

5.1. Fractional free-electron laser equation based on Caputo derivative

On setting $\lambda=0,\ \gamma=0,\ \nu=1$, above Example 4.1 reduces into following generalization of fractional free-electron laser equation based on Caputo derivative:

If $0<\mu<1$, $\mathfrak{Re}(\rho\delta-\beta)>0$, then FIDE that represents generalized FEL :

$$\left({}^{\mathsf{C}}\mathbf{D}^{\mu}_{0+}h\right)(t)+\alpha\int_0^t G_{\{\rho,\beta,\delta\}}(\omega,x,t)h(x)dx=0,\tag{65}$$

with $[h(t)]_{t=0+}=c$, has its solution in terms of LHGF as:

$$h(t)=c\sum_{k=0}^{\infty}(-\alpha)^k\left[G_{\{\rho,[(\beta-\mu)k-1],\delta k\}}(\omega,0,t)\right],\tag{66}$$

provided the sum on RHS of Eq. (66) converges.

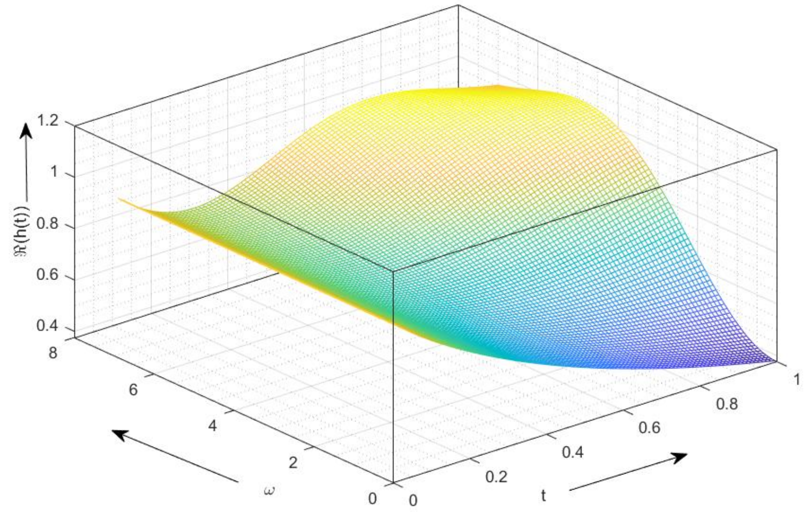

Figure 1: Graph demonstrates the real part of the solution for Caputo derivative of order $\mu = 1/9$

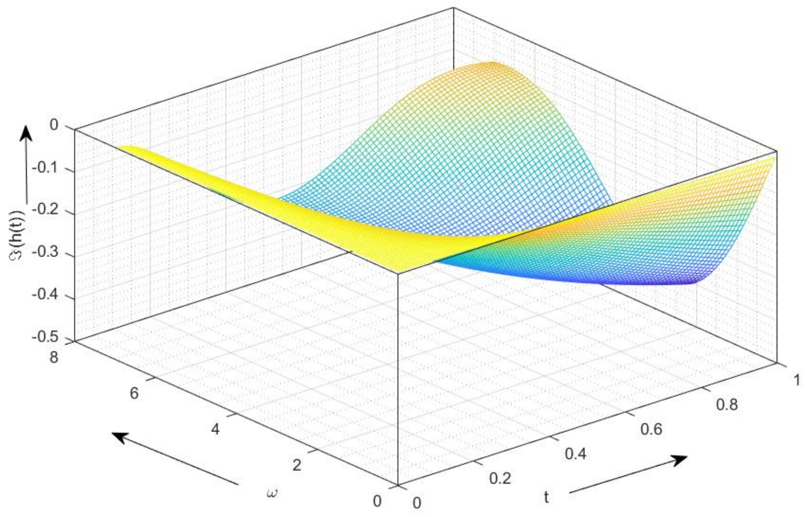

Figure 2: Graph exhibits the imaginary part of the solution for Caputo derivative of order $\mu = 1/9$

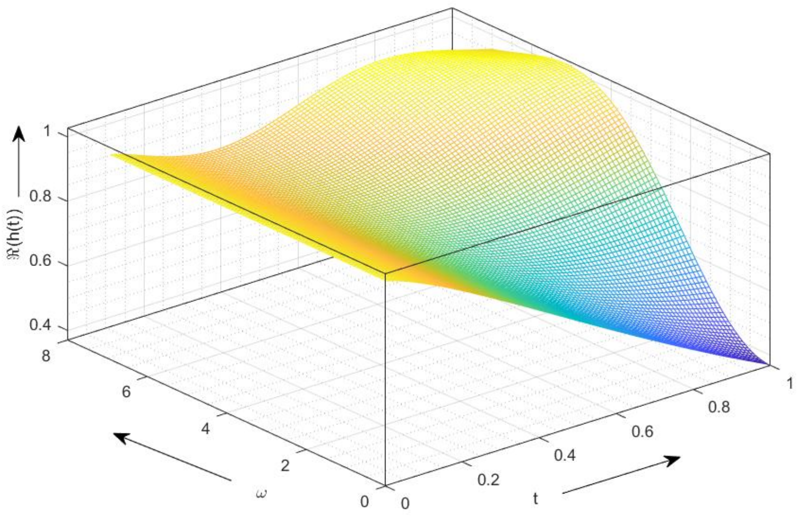

Figure 3: Graph displays the real part of the solution for Caputo derivative of order $\mu = 1/2$

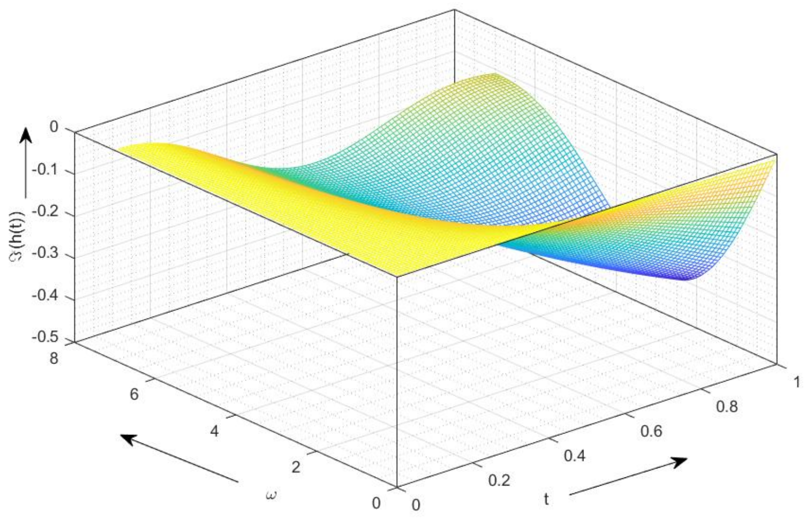

Figure 4: Graph describes the imaginary part of the solution for Caputo derivative of order $\mu = 1/2$

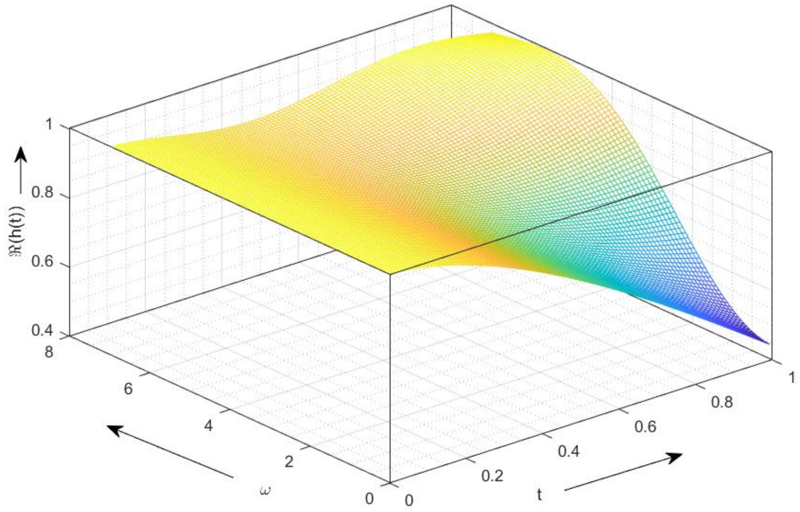

Figure 5: Graph indicates the real part of the solution for Caputo derivative of order $\mu = 8/9$

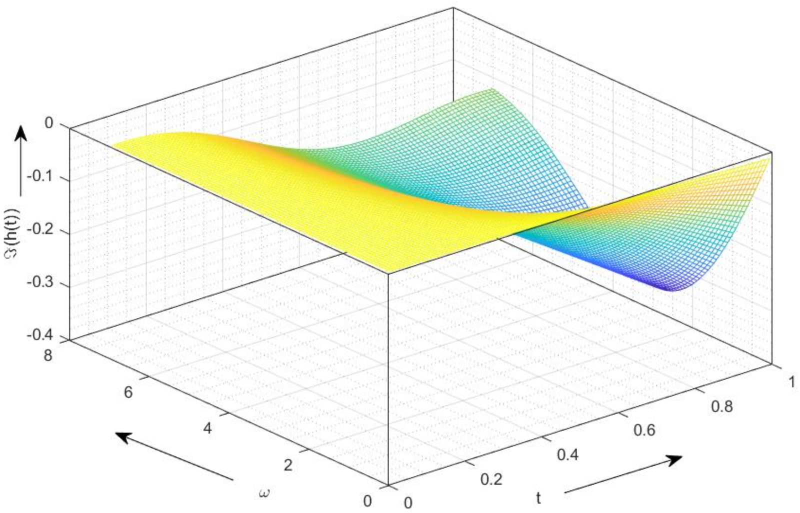

Figure 6: Graph reflects the imaginary part of the solution for Caputo derivative of order $\mu = 8/9$

5.2. Fractional free-electron laser equation based on Riemann-Liouville derivative

On substituting $\lambda = 0,\ \gamma = 0,\ \nu = 0$, above Example 4.1 give rise to the following fractional homogeneous fractional free-electron laser equation based on Riemann-Liouville derivative:

If $0 < \mu < 1$, $\mathfrak{Re}(\rho\delta - \beta) > 0$, then FIDE that represents generalized FEL:

$$\left({}^{\mathsf{RL}}\mathcal{D}^{\mu}_{0+}h\right)(t) + \alpha \int_0^t G_{\{\rho,\beta,\delta\}}(\omega, x, t)h(x)dx = 0, \tag{67}$$

with $\left({}^{\mathsf{RL}}\mathcal{I}^{(1-\mu)}_{0+}h(t)\right)_{t=0+} = c,$ has it solution in terms of LHGF as:

$$h(t) = c\sum_{k=0}^{\infty}(-\alpha)^k\Big[G_{\{\rho,[(\beta-\mu)k-\mu],\delta k\}}(\omega, 0, t)\Big], \tag{68}$$

provided the sum on RHS of Eq. (68) converges.

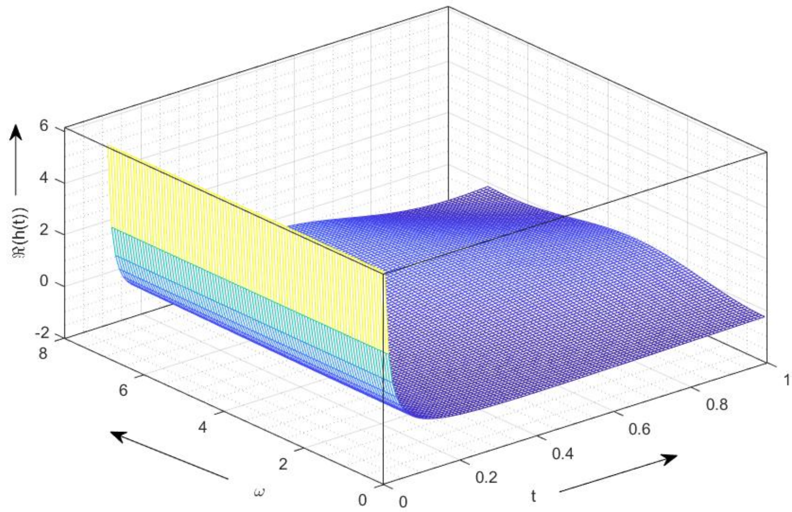

Figure 7: Graph demonstrates the real part of the solution for Riemann-Liouville derivative of order $\mu = 1/9$

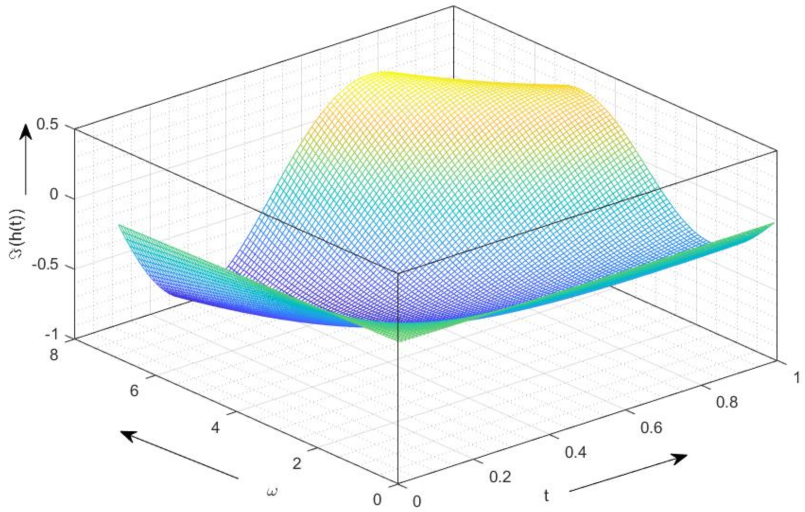

Figure 8: Graph exhibits the imaginary part of the solution for Riemann-Liouville derivative of order $\mu = 1/9$

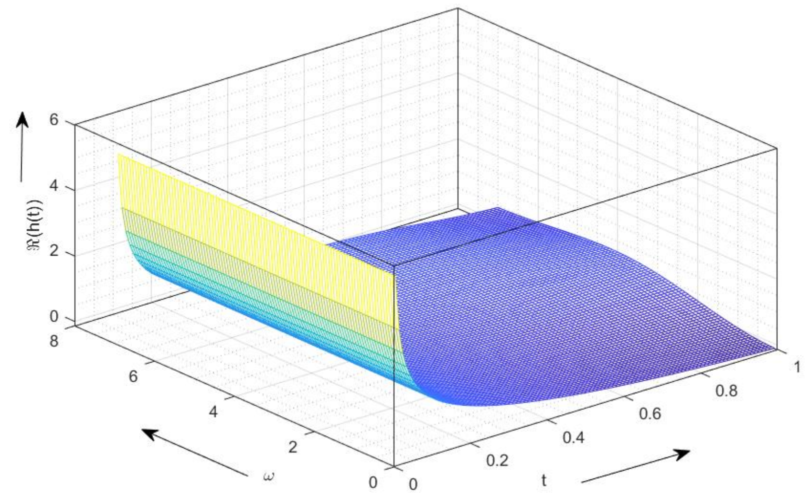

Figure 9: Graph displays the real part the of the solution for Riemann-Liouville derivative of order $\mu = 1/2$

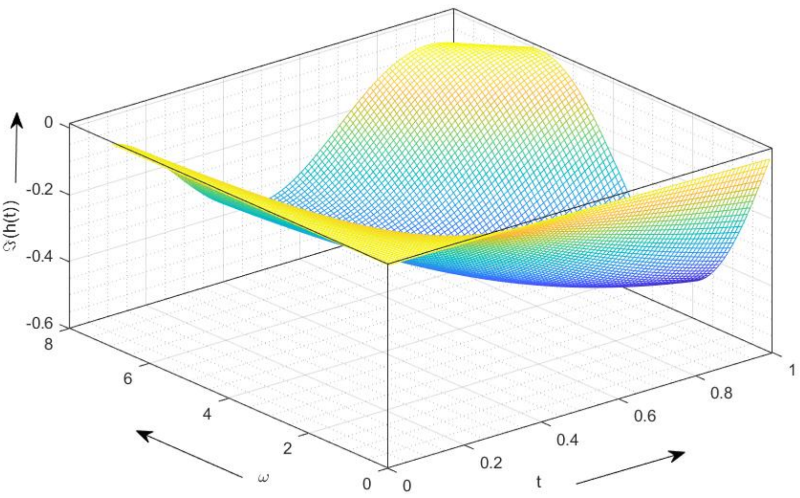

Figure 10: Graph describes the imaginary part of the solution for Riemann-Liouville derivative of order $\mu = 1/2$

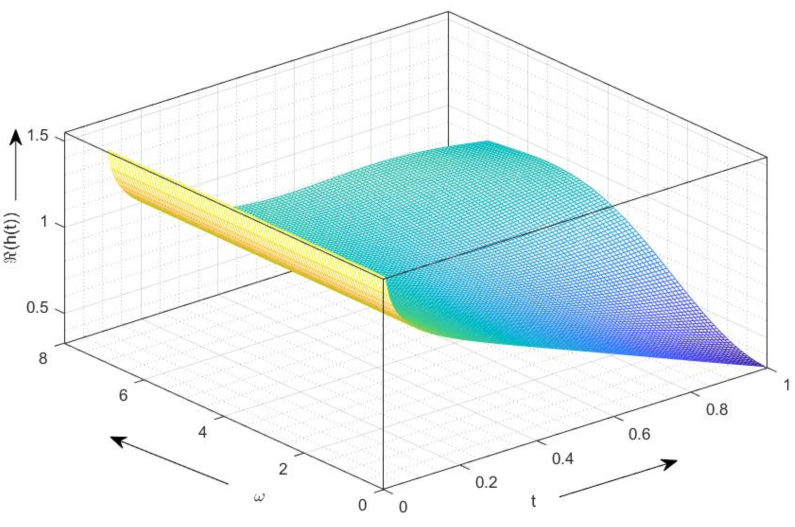

Figure 11: Graph indicates the real part the of the solution for Riemann-Liouville derivative of order $\mu = 8/9$

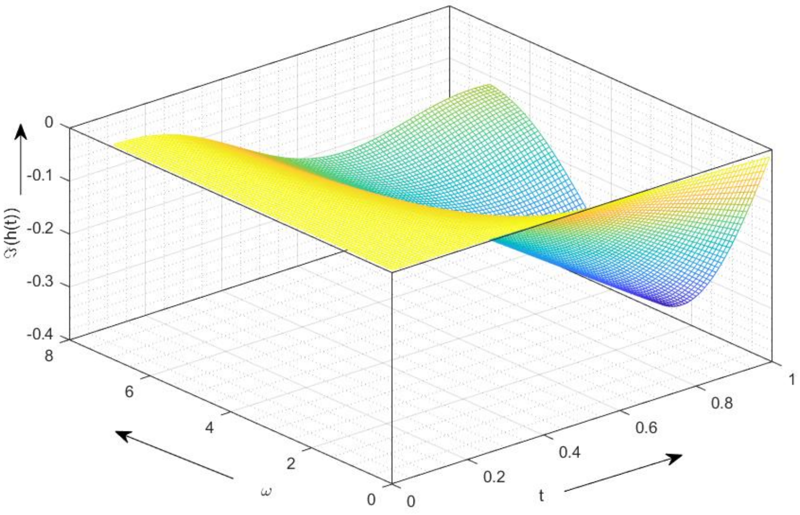

Figure 12: Graph reflects the imaginary part of the solution for Riemann-Liouville derivative of order $\mu = 8/9$

To illustrate the behaviour of the solutions of the above-mentioned generalized free-electron laser Equations (65) and (67) based on Caputo derivative and Riemann-Liouville derivative, respectively, the computation for the solutions are performed on MATLAB using series representation of LHGF. Particularly, the parameters for computations are taken as: $\alpha = 1, c = 1, \rho = 1, \delta = 1, (0 + i0.07) < \omega < (0 + i7)$, (with the difference of $(0 + i0.07)$) $0.01 < t < 1.0$, and $\beta = 0.2$ (with the difference of 0.01). The behavior of the obtained solutions are shown in figures. Figure 1 through Figure 6 exhibit the behaviour of the real and imaginary parts of the solutions for homogeneous fractional free-electron laser Eq. (65) with Caputo derivative. Figure 7 through Figure 12 demonstrate the behaviour of the real and imaginary parts of the solutions for homogeneous fractional free-electron laser Eq. (67) with Riemann-Liouville derivative.

6. Concluding remarks

In this paper, we have investigated a unified family of Volterra-type fractional integro-differential equations. The solution of the considered family is determined in the closed-forms of LHGF, which works well in case of increased time-domain complexity. To investigate the computational nature of the solution of the considered unified family, we have discussed the convergence of the solution. Several known and new fractional integro-differential equations involving different functions for fractional calculus in the kernel (obtained by proper choice of parameters in LHGF) accompanied by various forms of fractional derivatives, can be derived by specializing the parameters involved therein. Notably, the results can also be used to obtain closed-form solutions for several other Volterra-type fractional integro-differential equations that arise in different engineering sciences fields. From the application point of view, we have illustrated two forms of fractional free-electron laser and obtained their solutions in the closed and computable form of LHGF. Several graphical illustrations are presented, which demonstrate the behaviour of the solutions.

Remarkably, Hilfer-Prabhakar derivative interpolates between the Prabhakar derivative and its regularized version, given in Eq. (10) and Eq. (11), respectively. It can also be reduced into Hilfer derivative which may reduced into Riemann-Liouville and Caputo fractional derivatives by proper choice of parameters. Thus, the results established in the paper may be used to derive closed-form solutions for several Volterra-type fractional integro-differential equations, hitherto scattered in the literature.

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