On the Basis Property of Root Vectors Related to a Non-Self-Adjoint Analytic Operator and Applications

Hanen Ellouz, Ines Feki and Aref Jeribi

ABSTRACT: In the present paper, based on a separation condition on the spectrum of a self-adjoint operator T_0 on a separable Hilbert space $\mathcal{H}$, we prove that the system of root vectors of the perturbed operator $T(\varepsilon)$ given by

$$T(\varepsilon) := T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \ldots + \varepsilon^k T_k + \ldots$$

is complete and forms a basis with parentheses in $\mathcal{H}$, for small enough $|\varepsilon|$. Here $\varepsilon \in \mathbb{C}$ and $T_1, T_2, \ldots$ are linear operators on $\mathcal{H}$ having the same domain $\mathcal{D} \supset \mathcal{D}(T_0)$ and satisfying a specific growing inequality. The obtained results are of importance for applications to a non-self-adjoint Gribov operator in Bargmann space and to a non-self-adjoint problem deduced from a perturbation method for sound radiation.

AMS Subject Classification: 15A42, 47A55, 47B25, 46A35, 46B15, 65F15, 65H17.
Keywords and Phrases: Basis with parentheses; Elastic membrane; Gribov operator; Not condense; Spectrum.

1. Introduction

For non-self-adjoint perturbations of a self-adjoint operator, the crucial problem is the study of the spectral properties. For instance, the existence of a basis (possibly with parentheses) of root vectors is an important property. In order to prove the existence of such basis, several authors studied the comportment of the eigenvalues and established different conditions in terms of the spectrum (see [3]-[5], [8]-[13], [16]-[20], [22], [24] and [25]). Indeed, many non-self-adjoint ordinary differential operators

can be considered as a perturbation $T+B$ of a leading self-adjoint component T by its subordinate B. In [22], A. S. Markus claimed that $G = T+B$ admits an unconditional basis with parentheses of root vectors if B is p-subordinate to T and the eigenvalue-counting function of T satisfy a certain asymptotic growth condition. One might ask whether we can construct a basis if the p-subordinate condition is relaxed. A positive answer is given by A. A. Shkalikov [24]. He assumed that T is positive, self-adjoint with discrete spectrum and its eigenvalues $(\mu_n)_{n \in \mathbb{N}^*}$ are not condense, i.e.,

$$\mu_{n+q} - \mu_n \geq 1, \quad \text{for some} \quad q \in \mathbb{N}^*. \tag{1.1}$$

Further, he required that B verify

$$\|B\psi_n\| \leq b, \tag{1.2}$$

where $(\psi_n)_{n \in \mathbb{N}^*}$ is an orthonormal system of eigenvectors associated to the eigenvalues $(\mu_n)_{n \in \mathbb{N}^*}$ of T. Under these assumptions, he established an asymptotic relation between the eigenvalue-counting functions of G and T and he claimed that the system of root vectors of G forms a basis with parentheses in $\mathcal{H}$. More precisely, he proved the existence of a spectral condition

$$n(r, G) - n(r, T) = O(1),$$

under which he guarantees the existence of a basis with parentheses of root vectors (see [24, *Theorem* 2]).
Here $n(r, T)$ (respectively, $n(r, G)$) denotes the sum of multiplicities of all eigenvalues of T (respectively, G) contained in the disk $\{\lambda \in \mathbb{C} \quad \text{such that} \quad |\lambda| \leq r\}$.

Notice that in classical perturbation theorems for bases or Riesz bases, the authors always required that the eigenvalues of T are with multiplicity one (for instance, see [6], [7, *Theorem XIX*.2.7] and [21, *Theorem V*.4.15a]). Although, by assuming that the eigenvalues are with finite multiplicity, several authors such as A. Jeribi [18, 19], A. S. Markus [22], A. A. Shkalikov [24] and C. Wyss [25] proved the existence of bases with parentheses or unconditional bases with parentheses.
It is interesting to note here that the concept of bases (or unconditional bases) with parentheses is a natural generalization of the one of the bases (or Riesz bases).
Furthermore, [24, *Theorem* 2] ameliorates the result stated in [22]. Indeed, A. A. Shkalikov obtained a basis with parentheses under Eqs (1.1) and (1.2) which are much weaker.

Besides, in many situations, this result presents an important tool in the determining of the existence of bases. Among this direction we had the idea to exploit this outcome to study the Gribov operator (see [1], [2], [12] and [15]) originated from Reggeon field theory and constructed as a polynomial in the standard annihilation operator A and the standard creation operator A^*:

$$(A^*A)^3 + \varepsilon A^*(A + A^*)A + \varepsilon^2 (A^*A)^{3u_2} + ... + \varepsilon^k (A^*A)^{3u_k} + ...,$$

where $\varepsilon \in \mathbb{C}$ and $(u_k)_{k \in \mathbb{N}}$ is a strictly decreasing sequence with strictly positive terms such that $u_0 = 1$ and $u_1 = \frac{1}{2}$; while the expressions of the operators A and A^* are given by:

$$\begin{cases} A : \mathcal{D}(A) \subset \mathcal{B} \longrightarrow \mathcal{B} \\ \qquad\qquad \varphi \longrightarrow A\varphi(z) = \dfrac{d\varphi}{dz}(z) \\ \mathcal{D}(A) = \{\varphi \in \mathcal{B} \text{ such that } A\varphi \in \mathcal{B}\} \end{cases}$$

and

$$\begin{cases} A^* : \mathcal{D}(A^*) \subset \mathcal{B} \longrightarrow \mathcal{B} \\ \qquad\qquad \varphi \longrightarrow A^*\varphi(z) = z\varphi(z) \\ \mathcal{D}(A^*) = \{\varphi \in \mathcal{B} \text{ such that } A^*\varphi \in \mathcal{B}\}, \end{cases}$$

$$\mathcal{B} = \left\{ \varphi : \mathbb{C} \longrightarrow \mathbb{C} \text{ entire such that } \int_{\mathbb{C}} e^{-|z|^2} |\varphi(z)|^2 dz d\bar{z} < \infty \right\}.$$

Since $\{\varphi_n := \frac{z^n}{\sqrt{n!}}\}_{n \geq 1}$ is an orthonormal basis of eigenvectors of $(A^*A)^3$ associated to the eigenvalues $\{n^3\}_{n \geq 1}$, then we have

$$\left\| \left(\varepsilon A^* (A + A^*) A + \sum_{k=2}^{\infty} \varepsilon^k (A^*A)^{3u_k} \right) \varphi_n \right\| \leq \frac{|\varepsilon|}{1 - |\varepsilon|} (1 + 2\sqrt{2})(1 + n^3), \quad \text{for } |\varepsilon| < 1.$$

$$(1.3)$$

It is clear here that Eq. (1.3) does not verify Eq. (1.2). Consequently, [24, *Theorem* 2] can not be applied.

Further, if we consider the following integro-differential operator initially motivated by P. J. T. Filippi et al. [14] and deduced from a perturbation method for sound radiation (see also [8], [11] and [13]):

$$(I + \varepsilon K)^{-1} \frac{d^4\varphi}{dx^4} + \varepsilon (I + \varepsilon K)^{-1} K \left(\frac{d^4}{dx^4} - \left(\frac{d^4}{dx^4} \right)^{\frac{1}{2}} \right) \varphi = \lambda(\varepsilon)\varphi,$$

where K is the integral operator with kernel the Hankel function of the first kind and order 0 and ε is a complex number such that $|\varepsilon| < \frac{1}{\|K\|}$; we obtain

$$\left\| \sum_{k=1}^{\infty} (-1)^k \varepsilon^k K^k \frac{d^2\varphi_n}{dx^2} \right\| \leq \frac{|\varepsilon|}{1 - |\varepsilon|\|K\|} \|K\| \, \kappa n^4, \quad \text{for } |\varepsilon| < \frac{1}{\|K\|}.$$

Here $(\varphi_n)_{n \geq 1}$ denotes the system of eigenvectors of the operator

$$\begin{cases} \frac{d^4}{dx^4} : \mathcal{D}(\frac{d^4}{dx^4}) \subset L^2(]-L,L[) \longrightarrow L^2(]-L,L[) \\ \qquad\qquad\qquad \varphi \longrightarrow \dfrac{d^4\varphi}{dx^4} \\ \mathcal{D}(\frac{d^4}{dx^4}) = H_0^2(]-L,L[) \cap H^4(]-L,L[) \end{cases}$$

associated to the eigenvalues $(\lambda_n = \kappa n^4)_{n \geq 1}$ $(\kappa > 0)$. It is easy to check that $(\varphi_n)_{n \geq 1}$ forms an orthonormal basis of $L^2(]-L,L[)$.

Hence, Eq. (1.2) is not fulfilled and consequently [24, *Theorem* 2] can not be applied.

Among this direction and in order to overcome these bumps, we had the idea to extend [24, *Theorem* 2] to an abstract setting. More precisely, we continue the analysis started in [9] and we focus on the property of bases with parentheses of the analytic operator

$$T(\varepsilon) := T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \ldots + \varepsilon^k T_k + \ldots, \qquad (1.4)$$

where $\varepsilon \in \mathbb{C}$, T_0 is a closed linear densely defined operator on a separable Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(T_0)$ while $T_1, T_2, \ldots$ are linear operators on $\mathcal{H}$ having the same domain $\mathcal{D} \supset \mathcal{D}(T_0)$ and satisfying

$$\|T_k \varphi\| \leq q^{k-1}(a\|\varphi\| + b\|T_0\varphi\|^{\beta}\|\varphi\|^{1-\beta})$$

for all $\varphi \in \mathcal{D}(T_0)$ and for all $k \geq 1$, where $\beta \in]0, \frac{1}{2}[$ and a, b and $q > 0$.

We would like to mention here that the perturbed operator (1.4) was introduced by B. Sz. Nagy in [23] and considered later in some valuable papers such as [3], [5] and [8]-[13].

Furthermore, it is interesting to note here that in [9] we derived a precise description to the localization of the spectrum of the perturbed operator (1.4) and we proved an asymptotic relation between the eigenvalue-counting functions of T_0 and $T(\varepsilon)$. In other words, we claimed that the difference between the eigenvalue-counting functions of T_0 and $T(\varepsilon)$ is bounded by a constant. This generalization is of great importance. In fact, it allows us to control the jump of the eigenvalue-counting function of some analytic operators where the criteria of A. A. Shkalikov [24] can not be applied.

Now, based on the asymptotic relation between the eigenvalue-counting functions of T_0 and $T(\varepsilon)$ developed in [9], can we construct a basis with parentheses of root vectors of the perturbed operator $T(\varepsilon)$? Indeed, in view of [9, *Proposition* 3.1] the spectrum of $T(\varepsilon)$ is discrete for $|\varepsilon| < \frac{1}{q+\beta b}$. So, we consider $E_n = \cup_{m \geq 1} N(T(\varepsilon) - \lambda_n(\varepsilon))^m$ the root linear finite dimensional subspace whose dimension is called algebraic multiplicity of the eigenvalue $\lambda_n(\varepsilon)$. These subspaces are linearly independent and vectors in E_n are called root vectors of $T(\varepsilon)$. Following some ideas due to A. A. Shkalikov [24], we prove first that the system of root vectors of the perturbed operator $T(\varepsilon)$ is complete. Notice that our result improves Theorem 4.3 stated in [12]. In fact, not only the assumptions used in [12] are relaxed but also the values that takes $|\varepsilon|$ are greater than the one considered in [12, *Theorem* 4.3]. Furthermore, it can be considered as an extension of [24, *Lemma* 7] to an analytic operator.

Having obtained this aforementioned result, one might seek if it forms a basis with parentheses. Actually, using the spectral condition developed in [9], we prove that for $|\varepsilon|$ enough small, the system of root vectors of $T(\varepsilon)$ forms a basis with parentheses in $\mathcal{H}$.

We point out here that our result ameliorates [13, *Theorem* 3.4] since they established the existence of Riesz basis using a spectral analysis method based on the fact that the eigenvalues of T_0 are with multiplicity one; while we investigate the existence of basis with parentheses by supposing that the eigenvalues are with finite multiplicity. Further, our result might be regarded as an extension of [24, *Theorem* 2]. In fact, we guarantee the existence of basis with parentheses for some analytic operators where Eq. (1.2) considered by A. A. Shkalikov in [24] is not verified.

The present paper consists of four sections: In section 2, we introduce some basic definitions and auxiliary results connected to the main body of the paper. Section 3 is devoted to prove the completeness of the system of root vectors of $T(\varepsilon)$ and the existence of basis with parentheses of root vectors. In the last section, we apply the obtained results to a Gribov operator in Bargmann space and to a problem of radiation of a vibrating structure in a light fluid.

2. Preliminaries

In order to state our main results, let us begin with some definitions and preliminary results that we will need in the sequel. For this, let us consider a Hilbert space $\mathcal{H}$.

Definition 2.1. [22, p. 16] Let A be a linear operator such that its resolvent set, $\rho(A)$, is not empty. An operator B is said to be A-compact if its domain $\mathcal{D}(B)$ contains $\mathcal{D}(A)$ and if the operator $BR_\lambda(A)$ is compact, where $\lambda \in \rho(A)$. $\diamond$

Definition 2.2. Let K be a compact operator on $\mathcal{H}$. K is said to belong to the Carleman-class $\mathcal{C}_p$ $(p > 0)$, if the series $\sum_{n=1}^{\infty}[s_n(\sqrt{K})]^p$ converges, where $s_n(\sqrt{K}), n = 1, 2, \ldots$, are the eigenvalues of the operator $\sqrt{K^*K}$. $\diamond$

Definition 2.3. [22, p. 18] An operator K is said to be of finite order if it belongs to the Carleman-class $\mathcal{C}_p$ $(p > 0)$. $\diamond$

Markus's theorem is formulated as:

Theorem 2.1. [*22, Theorem 4.3*] *Let A be a normal operator whose resolvent belongs to the Carleman-class $\mathcal{C}_p$ $(p > 0)$, and whose spectrum lies on a finite number of rays $\arg \lambda = \alpha_k(k = 1, \ldots, n)$. If B is A-compact, then the operator $G = A + B$ has a compact resolvent and the system of its root vectors is complete in $\mathcal{H}$.* $\diamond$

Lemma 2.1. [*24, Lemma 8*] *Let $F(\lambda)$ be a scalar meromorphic function with finite order in an angle $\Lambda_\alpha = \{\lambda : |\arg \lambda| < \alpha\}$ and the poles of $F(\lambda)$ in this angle lie inside the strip $|Im\lambda| \leq h$, $h > 0$. Suppose that $|F(\lambda)| \leq M$ on the half-lines $Im\lambda = \pm(h + \delta)$, $\delta > 0$, inside the angle Λ_α. Then the following estimate holds inside the strip $|Im\lambda| \leq h + \delta$ as $Re\lambda \to \infty$ outside an exceptional set of disks $\mathcal{D}$:*

$$\ln|F(\lambda)| \leq C \left(M + \sup_{|t-r|\leq r^\eta} (n(t+1, F) - n(t, F)) \right), \qquad r = |\lambda|$$

where $n(t, F)$ is the pole-counting function for F and the number η can be taken arbitrarily small. For any $d > 0$, the exceptional set of disks $\mathcal{D}$ can be chosen in such a way that the total sum of the radii of the disks from $\mathcal{D}$ inside the rectangle $|Im\lambda| \leq h$, $t \leq Re\lambda \leq t+1$ does not exceed d for any sufficiently large t. The constant C depends on δ, η, and d (the dependence on d is proportional to $\ln d$) but does not depend on r and F. $\diamond$

In the remaining part of this section, we introduce the concept of basis (possibly with parentheses).

Definition 2.4. [22, *p. 25*] A sequence $\{V_n\}_{n=1}^{\infty}$ of subspaces of a Hilbert space $\mathcal{H}$ is called a basis (of subspaces), if any vector belonging to $\mathcal{H}$ can be uniquely represented as a series

$$\varphi = \sum_{n=1}^{\infty} \varphi_n \quad \text{such that} \quad \varphi_n \in V_n. \qquad \diamondsuit$$

Definition 2.5. [22, *p. 27*] A linearly independent sequence $\{\varphi_n\}_{n=1}^{\infty}$ is called a basis with parentheses for a Hilbert space $\mathcal{H}$, if there exists a sequence of positive integers $(n_k)_k$ such that $n_0 = 1$ and the subspaces spanned by the vectors $\{\varphi_n\}_{n_{k-1}}^{n_k-1}$ form a basis for $\mathcal{H}$. $\qquad \diamondsuit$

Theorem 2.2. [22, *Lemma* 6.1] *Let* $\{P_k\}_{k=1}^{\infty}$ *be a sequence of disjoint projections* (*i.e.,*
$P_j P_k = \delta_{jk} P_k$). *If the sequence of subspaces* $R_k = Im P_k \quad (k \in \mathbb{N}^*)$ *is complete in* $\mathcal{H}$, *then it is a basis for* $\mathcal{H}$ *if and only if*

$$\sup_n \left\| \sum_{k=1}^{n} P_k \right\| < \infty. \qquad \diamondsuit$$

3. Main results

Let $\mathcal{H}$ be a separable Hilbert space and T_0 be a linear operator on $\mathcal{H}$ verifying the following hypotheses:
($H1$) T_0 is self-adjoint, positive and with domain $\mathcal{D}(T_0)$ in $\mathcal{H}$.
($H2$) The resolvent of T_0 is compact and its eigenvalues $(\lambda_n)_n$ verify

$$\lambda_{n+p} - \lambda_n \geq 1 \text{ for some } p \in \mathbb{N}^*.$$

Let $T_1, T_2, T_3, \dots$ be linear operators on $\mathcal{H}$ having the same domain $\mathcal{D}$ and satisfying the hypothesis:
($H3$) $\mathcal{D} \supset \mathcal{D}(T_0)$ and there exist $a, b, q > 0$ and $\beta \in]0, \frac{1}{2}[$ such that for all $k \geq 1$

$$\|T_k \varphi\| \leq q^{k-1}(a\|\varphi\| + b\|T_0\varphi\|^{\beta}\|\varphi\|^{1-\beta}) \text{ for all } \varphi \in \mathcal{D}(T_0).$$

Let ε be a non zero complex number and consider the eigenvalue problem

$$\begin{cases} T_0\varphi + \varepsilon T_1\varphi + \varepsilon^2 T_2\varphi + \cdots + \varepsilon^k T_k\varphi + \cdots = \lambda\varphi \\ \varphi \in \mathcal{D}(T_0). \end{cases}$$

Before stating our main results, we shall recall the following theorem.

Theorem 3.1. [12, *Theorem* 2.1] *Suppose that hypotheses* ($H1$) *and* ($H3$) *hold. Then for* $|\varepsilon| < q^{-1}$, *the series* $\sum_{i \geq 0} \varepsilon^i T_i \varphi$ *converges for all* $\varphi \in \mathcal{D}(T_0)$. *If* $T(\varepsilon)\varphi$ *denotes its limit, then* $T(\varepsilon)$ *is a linear operator with domain* $\mathcal{D}(T_0)$ *and for* $|\varepsilon| < (q + \beta b)^{-1}$, *the operator* $T(\varepsilon)$ *is closed.* $\qquad \diamondsuit$

3.1. Completeness of the system of root vectors of $T(\varepsilon)$

The aim of this part is to establish the completeness of the system of root vectors of the perturbed operator $T(\varepsilon)$ in $\mathcal{H}$.

To this end, we need first to recall the following proposition developed in [9].

Taking into account Theorem 3.1, we denote by $B(\varepsilon) := \sum_{k=1}^{\infty} \varepsilon^k T_k$.

Proposition 3.1. *[9, Proposition 3.1] Assume that hypotheses* (H1)-(H3) *hold. Then, for* $|\varepsilon| < \frac{1}{q+\beta b}$, *the operator* $B(\varepsilon)$ *is* T_0-compact. *Moreover, the operator* $T(\varepsilon)$ *is with compact resolvent.* $\diamondsuit$

Now, we are ready to state our result.

Theorem 3.2. *Assume that hypotheses* (H1)-(H3) *are verified. Then, for* $|\varepsilon| < \frac{1}{q+\beta b}$, *the system of root vectors of the operator* $T(\varepsilon)$ *is complete in* $\mathcal{H}$. $\diamondsuit$

Remark 3.1.

(i) Theorem 3.2 extends [24, *Lemma* 7] to an analytic operator instead of the sum of two operators. Besides, we have proved that the system of root vectors of $T(\varepsilon)$ is complete even if the criteria of A. A. Shkalikov (Eq. (1.2)) is not satisfied.

(ii) Theorem 3.2 ameliorates Theorem 4.3 stated in [12]. Indeed, in order to guarantee that the operator $B(\varepsilon)$ is T_0-compact, the authors in [12] assumed that T_k is T_0-compact for all $k \geq 1$; whereas Proposition 3.1 ensure this result without this assumption. On the other hand, the values of $|\varepsilon|$ for which the system of root vectors of the operator $T(\varepsilon)$ is complete in $\mathcal{H}$, are greater than the one considered in [12, *Theorem* 4.3]. $\diamondsuit$

Proof of Theoerm 3.2.

In view of hypotheses (H1) and (H2), we have T_0 is self-adjoint with compact resolvent. Further, it follows from hypothesis (H2) that

$$\lambda_{n+1} - \lambda_1 = \underbrace{\lambda_{n+1} - \lambda_{(n+1)-p}}_{\geq 1} + \underbrace{\lambda_{(n+1)-p} - \lambda_{(n+1)-2p}}_{\geq 1} + \dots$$
$$+ \underbrace{\lambda_{1+p} - \lambda_{(n+1)-\frac{n}{p}p(=1)}}_{\geq 1} \geq \frac{n}{p}. \tag{3.1}$$

Thus, Eq. (3.1) yields $\lambda_n \geq \frac{n-1}{p} + \lambda_1$. So, there exists $P > 1$ such that the series $\sum_{n \geq 1} (\frac{1}{\lambda_n})^P$ is convergent. Consequently, the resolvent of T_0 belongs to the Carleman-class $\mathcal{C}_P$. Moreover, in virtue of Proposition 3.1, the operator $B(\varepsilon)$ is T_0-compact for $|\varepsilon| < \frac{1}{q+\beta b}$. Consequently, Theorem 2.1 implies that for $|\varepsilon| < \frac{1}{q+\beta b}$, the system of root vectors of the operator $T(\varepsilon)$ is complete in $\mathcal{H}$. $\square$

Corollary 3.1. *Suppose that hypotheses (H1) and (H3) are verified. Moreover, assume that*

$$\lambda_{n+p}^{1-\alpha} - \lambda_n^{1-\alpha} \geq 1, \quad where \ \ 0 \leq \alpha < 1. \tag{3.2}$$

Hence, for $\beta \in]0, 1 + \frac{\alpha-1}{2}[$ and $|\varepsilon| < \frac{1}{q+\beta b}$ the system of root vectors of the operator $T(\varepsilon)$ is complete in $\mathcal{H}$. ◇

Proof. It follows from Eq. (3.2) that $\lambda_n^{1-\alpha} \geq \frac{n-1}{p} + \lambda_1^{1-\alpha}$. Hence, there exists $P > 1 - \alpha$ such that the series $\sum_{n \geq 1} (\frac{1}{\lambda_n})^P$ is convergent. As T_0 is self-adjoint with compact resolvent, then the resolvent of T_0 belongs to the Carleman-class $\mathcal{C}_P$. Further, due to [9, *Corollary* 3.1] the operator $B(\varepsilon)$ is T_0-compact for $|\varepsilon| < \frac{1}{q+\beta b}$. Hence, according to Theorem 2.1, we deduce that the system of root vectors of the operator $T(\varepsilon)$ is complete in $\mathcal{H}$ for $|\varepsilon| < \frac{1}{q+\beta b}$. □

3.2. Basis with parentheses of root vectors of $T(\varepsilon)$

In Theorem 3.2, we have proved that the system of root vectors of the operator $T(\varepsilon)$ is complete. The question that occurs is whether this system forms a basis in $\mathcal{H}$. In other words, if

$$P_{n,\varepsilon} = \int_{\partial \Delta_n} (\lambda - T(\varepsilon))^{-1} d\lambda$$

denotes the spectral projection corresponding to the spectrum of $T(\varepsilon)$ inside Δ_n where Δ_n is a bounded closed isolated part of the spectrum of $T(\varepsilon)$, then the series $\sum_n P_{n,\varepsilon} f$ is convergent and its sum is it f.

To answer to this question, we shall prove some preliminary results.

Lemma 3.1. *Let τ be an arbitrary positive number. If $|Im\lambda| \geq \tau$, then for $|\varepsilon| < \frac{1}{q}$ there exists a positive number $N(\varepsilon, a, p, q, \tau)$ such that*

$$\sum_{n=1}^{\infty} \frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \lambda_n|^2} < N(\varepsilon, a, p, q, \tau). \tag{3.3}$$

If $Re\lambda \leq -\tau$, then for $|\varepsilon| < \frac{1}{q}$ there exists also a positive number $N_1(\varepsilon, a, p, q, \tau)$ such that

$$\sum_{n=1}^{\infty} \frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \lambda_n|^2} < N_1(\varepsilon, a, p, q, \tau). \tag{3.4}$$

◇

Proof. Let $n \in \mathbb{N}^*$ and λ_n be the eigenvalue number n of T_0. We have

$$\begin{aligned}
\|B(\varepsilon)\varphi_n\| &= \|(\varepsilon T_1 + \varepsilon^2 T_2 + \ldots)\varphi_n\| \\
&\leq \sum_{i=1}^{\infty} \|\varepsilon^i T_i \varphi_n\|.
\end{aligned}$$

Then, in view of hypothesis $(H3)$ we obtain

$$
\begin{aligned}
\|B(\varepsilon)\varphi_n\| &\leq \sum_{i=1}^{\infty} |\varepsilon|^i q^{i-1} (a\|\varphi_n\| + b\|T_0\varphi_n\|^\beta \|\varphi_n\|^{1-\beta}) \\
&\leq \sum_{i=1}^{\infty} |\varepsilon|^i q^{i-1} (a + b\lambda_n^\beta).
\end{aligned}
\tag{3.5}
$$

Hence, for $|\varepsilon| < \frac{1}{q}$ it follows from Eq. (3.5) that

$$
\frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \lambda_n|^2} \leq \frac{|\varepsilon|^2}{(1 - |\varepsilon|q)^2} \left(\frac{a^2}{|\lambda - \lambda_n|^2} + \frac{2ab\lambda_n^\beta}{|\lambda - \lambda_n|^2} + \frac{b^2\lambda_n^{2\beta}}{|\lambda - \lambda_n|^2} \right).
\tag{3.6}
$$

Now, let $\sigma = Re\lambda$ and $\lambda = \sigma \pm i\tau$, where $\tau > 0$. So, there exists $k \in \mathbb{N}^*$ such that $\lambda_{k-1} \leq \sigma$ and $\lambda_k > \sigma$. Thus, we have

$$
\lambda_k - \sigma > \lambda_k^\beta \left(\lambda_k^{1-\beta} - \sigma^{1-\beta} \right) \geq C_1 \lambda_k^\beta \quad (C_1 > 0)
\tag{3.7}
$$

and

$$
|\lambda - \lambda_{k-1}| \geq ||\lambda| - \lambda_{k-1}| > |\lambda|^\beta \left(|\lambda|^{1-\beta} - \lambda_{k-1}^{1-\beta} \right) \geq C_2 \lambda_{k-1}^\beta \quad (C_2 > 0).
\tag{3.8}
$$

Then, Eqs (3.7), (3.8) imply that

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{\lambda_n^{2\beta}}{|\lambda_n - \sigma|^2} &< \frac{1}{C_1^2} + \frac{1}{C_2^2} + \sum_{n<k-1} \frac{\lambda_n^{2\beta}}{|\lambda_n - \sigma|^2} + \sum_{n>k} \frac{\lambda_n^{2\beta}}{|\lambda_n - \sigma|^2} \\
&\leq \frac{2}{C^2} + \sum_{n<k-1} \frac{\lambda_n^{2\beta}}{|\lambda_n - \sigma|^2} + \sum_{n>k} \frac{\lambda_n^{2\beta}}{|\lambda_n - \sigma|^2},
\end{aligned}
\tag{3.9}
$$

where $C := \min\{C_1, C_2\}$. Further, since $\beta \in]0, \frac{1}{2}[$, hence for $n < k-1$ we obtain

$$
\begin{aligned}
\sigma - \lambda_n &> \lambda_{k-1} - \lambda_n \\
&> (1-\beta)\lambda_n^\beta (\lambda_{k-1} - \lambda_n)^{1-\beta} (\lambda_{k-1} - \lambda_n)^\beta \lambda_n^{-\beta} \\
&\geq \gamma_1(1-\beta)\lambda_n^\beta (\lambda_{k-1} - \lambda_n)^{1-\beta}, \quad 0 < \gamma_1 < 1
\end{aligned}
\tag{3.10}
$$

and for $n > k$ we have

$$
\begin{aligned}
\lambda_n - \sigma &> \lambda_n - \lambda_k \\
&> (1-\beta)\lambda_n^\beta (\lambda_n - \lambda_k)^{1-\beta} (\lambda_n - \lambda_k)^\beta \lambda_n^{-\beta} \\
&\geq \gamma_2(1-\beta)\lambda_n^\beta (\lambda_n - \lambda_k)^{1-\beta}, \quad 0 < \gamma_2 < 1.
\end{aligned}
\tag{3.11}
$$

So, Eqs (3.9), (3.10) and (3.11) yield

$$\sum_{n=1}^{\infty} \frac{\lambda_n^{2\beta}}{|\lambda_n - \sigma|^2} < \frac{2}{C^2} + \sum_{n<k-1} \frac{1}{\gamma_1^2(1-\beta)^2 \left(\lambda_{k-1} - \lambda_n\right)^{2(1-\beta)}}$$
$$+ \sum_{n>k} \frac{1}{\gamma_2^2(1-\beta)^2 \left(\lambda_n - \lambda_k\right)^{2(1-\beta)}}.$$

Consequently, if we put $\gamma := \min\{\gamma_1, \gamma_2\}$, we obtain

$$\sum_{n=1}^{\infty} \frac{\lambda_n^{2\beta}}{|\lambda_n - \sigma|^2} \quad < \quad \frac{2}{C^2} + \sum_{n<k-1} \frac{1}{\gamma^2(1-\beta)^2 \left(\lambda_{k-1} - \lambda_n\right)^{2(1-\beta)}} +$$
$$\sum_{n>k} \frac{1}{\gamma^2(1-\beta)^2 \left(\lambda_n - \lambda_k\right)^{2(1-\beta)}}. \tag{3.12}$$

As $\lambda_n \geq \frac{n-1}{p} + \lambda_1$, Eq. (3.12) yields

$$\sum_{n=1}^{\infty} \frac{\lambda_n^{2\beta}}{|\lambda - \lambda_n|^2} < \sum_{n=1}^{\infty} \frac{\lambda_n^{2\beta}}{|\lambda_n - \sigma|^2}$$

$$< \frac{2}{C^2} + \sum_{n<k-1} \frac{p^{2(1-\beta)}}{\gamma^2(1-\beta)^2 \left(k-1-n\right)^{2(1-\beta)}} + \sum_{n>k} \frac{p^{2(1-\beta)}}{\gamma^2(1-\beta)^2 \left(n-k\right)^{2(1-\beta)}}$$

$$\leq \frac{2}{C^2} + \sum_{n<k-1} \frac{p^{2(1-\beta)}}{\gamma^2(1-\beta)^2 \left(k-1-n\right)^{2(1-\beta)}} + \sum_{m=1}^{\infty} \frac{p^{2(1-\beta)}}{\gamma^2(1-\beta)^2 m^{2(1-\beta)}}$$

$$< \frac{2}{C^2} + \frac{2p^{2(1-\beta)}}{\gamma^2(1-\beta)^2} \sum_{m=1}^{\infty} \frac{1}{m^{2(1-\beta)}} =: \xi_1 < \infty.$$

Moreover, if we use the same argument as above with $\frac{\beta}{2}$ we get

$$\sum_{n=1}^{\infty} \frac{\lambda_n^{\beta}}{|\lambda - \lambda_n|^2} < \frac{2}{C^2} + \frac{8p^{2-\beta}}{\gamma^2(2-\beta)^2} \sum_{m=1}^{\infty} \frac{1}{m^{2-\beta}} =: \xi_2 < \infty.$$

Consequently, the series $\sum_n \frac{2ab\lambda_n^{\beta}}{|\lambda-\lambda_n|^2}$ and $\sum_n \frac{b^2\lambda_n^{2\beta}}{|\lambda-\lambda_n|^2}$ are convergent. So, there exists $\xi > 0$ verifying

$$\sum_{n=1}^{\infty} \frac{2ab\lambda_n^{\beta}}{|\lambda - \lambda_n|^2} + \sum_{n=1}^{\infty} \frac{b^2\lambda_n^{2\beta}}{|\lambda - \lambda_n|^2} < b^2\xi_1 + 2ab\xi_2 =: \xi. \tag{3.13}$$

To complete the proof of our result, we follow some ideas due to [24].

• Let us consider $|Im\lambda| \geq \tau$. For $|\varepsilon| < \frac{1}{q}$, it follows from Eqs (3.6) and (3.13) that

$$\sum_{n=1}^{\infty} \frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \lambda_n|^2} \quad < \frac{|\varepsilon|^2}{(1 - |\varepsilon|q)^2} \left(\xi + a^2 \sum_{n=1}^{\infty} \frac{1}{(\lambda_n - \sigma)^2 + \tau^2} \right).$$

On the other hand, hypothesis $(H2)$ implies that

$$\lambda_{k+j+sp} - \sigma \geq s \quad \text{and} \quad \sigma - \lambda_{k-j-sp-1} \geq s, \quad \text{where} \quad j = 0, \ldots, p-1 \text{ and } s = 0, 1, \ldots.$$

Hence, for $|\varepsilon| < \frac{1}{q}$ we obtain

$$\sum_{n=1}^{\infty} \frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \lambda_n|^2} < \frac{|\varepsilon|^2}{(1 - |\varepsilon|q)^2} \left(\xi + a^2 p \left(\sum_{s=0}^{\frac{k}{p}-1} \frac{1}{s^2 + \tau^2} + \sum_{s=0}^{\infty} \frac{1}{s^2 + \tau^2} \right) \right).$$

So, for $|\varepsilon| < \frac{1}{q}$ we have

$$\sum_{n=1}^{\infty} \frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \lambda_n|^2} < \frac{|\varepsilon|^2}{(1 - |\varepsilon|q)^2} \left(\xi + 2pa^2 \left(\sum_{s=1}^{\infty} \frac{1}{s^2 + \tau^2} + \frac{1}{\tau^2} \right) \right)$$

$$< \frac{|\varepsilon|^2}{(1 - |\varepsilon|q)^2} \left(\xi + 2pa^2 \left(\frac{1}{\tau^2} + \int_0^{\infty} \frac{dx}{x^2 + \tau^2} \right) \right)$$

$$\leq N(\varepsilon, a, p, q, \tau),$$

where

$$N(\varepsilon, a, p, q, \tau) := \frac{|\varepsilon|^2}{(1 - |\varepsilon|q)^2} \left(\xi + a^2 \frac{p}{\tau} \left(\pi + \frac{2}{\tau} \right) \right).$$

- Now, if $Re\lambda \leq -\tau$. It follows from hypothesis $(H2)$ that

$$\lambda_{1+j+sp} - \sigma \geq s - \sigma \geq s + \tau, \quad \text{where } j = 0, \ldots, p-1 \text{ and } s = 0, 1, \ldots, \quad (3.14)$$

since $\lambda_{1+j} > 0$ and $\lambda_{1+j+sp} - \sigma \geq s + \lambda_{1+j} - \sigma$. So, Eqs (3.6), (3.13) and (3.14) imply that for $|\varepsilon| < \frac{1}{q}$ we have

$$\sum_{n=1}^{\infty} \frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \lambda_n|^2} < \frac{|\varepsilon|^2}{(1 - |\varepsilon|q)^2} \left(\xi + a^2 \sum_{n=1}^{\infty} \frac{1}{|\sigma - \lambda_n|^2} \right)$$

$$\leq \frac{|\varepsilon|^2}{(1 - |\varepsilon|q)^2} \left(\xi + pa^2 \left(\sum_{s=0}^{\infty} \frac{1}{(\tau + s)^2} \right) \right)$$

$$< \frac{|\varepsilon|^2}{(1 - |\varepsilon|q)^2} \left(\xi + pa^2 \left(\int_0^{\infty} \frac{dx}{(\tau + x)^2} + \frac{1}{\tau^2} \right) \right)$$

$$\leq N_1(\varepsilon, a, p, q, \tau),$$

where

$$N_1(\varepsilon, a, p, q, \tau) := \frac{|\varepsilon|^2}{(1 - |\varepsilon|q)^2} \left(\xi + a^2 \frac{p}{\tau} \left(1 + \frac{1}{\tau} \right) \right). \qquad \square$$

The following proposition holds (see [24]).

Proposition 3.2. *We have*

$$\|B(\varepsilon)(\lambda - T_0)^{-1}\|^2 \leq \sum_{n=1}^{\infty} \frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \lambda_n|^2}.$$

$\diamond$

We denote by $S_h := \{\lambda \text{ such that } |Im\lambda| < h \text{ and } Re\lambda > -h\}$, with $h > 0$ (see Figure 1).
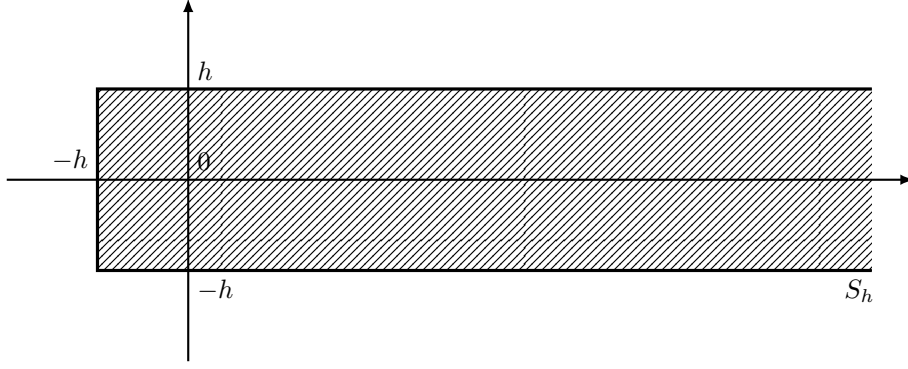

Figure 1

Proposition 3.3. *For small enough $|\varepsilon|$, the spectrum of the operator $T(\varepsilon)$ lies in the half-strip S_h.* $\diamond$

Proof. Let $\lambda \in \mathbb{C}$ such that $|Im\lambda| \geq h$ or $Re\lambda \leq -h$. Since T_0 is self-adjoint and positive, then we have

$$\lambda - T(\varepsilon) = [I - B(\varepsilon)(\lambda - T_0)^{-1}](\lambda - T_0). \tag{3.15}$$

Further, combining Eq. (3.3) together with Proposition 3.2, we obtain for $|Im\lambda| \geq h$ and $|\varepsilon| < \frac{1}{q}$

$$\|B(\varepsilon)(\lambda - T_0)^{-1}\|^2 < N(\varepsilon, a, p, q, h). \tag{3.16}$$

So, for $|Im\lambda| \geq h$ and $|\varepsilon| < \frac{1}{q + \sqrt{\xi + a^2 \frac{p}{h}(\pi + \frac{2}{h})}}$ we get

$$\|B(\varepsilon)(\lambda - T_0)^{-1}\| < 1.$$

On the other hand, for $Re\lambda \leq -h$ and $|\varepsilon| < \frac{1}{q}$, Eq. (3.4) and Proposition 3.2 yield

$$\|B(\varepsilon)(\lambda - T_0)^{-1}\|^2 < N_1(\varepsilon, a, p, q, h). \tag{3.17}$$

Hence, for $Re\lambda \leq -h$ and $|\varepsilon| < \frac{1}{q + \sqrt{\xi + a^2 \frac{p}{h}(1 + \frac{1}{h})}}$ we obtain

$$\|B(\varepsilon)(\lambda - T_0)^{-1}\| < 1.$$

Consequently, for

$$|\varepsilon| \quad < \min\left\{ \frac{1}{q + \sqrt{\xi + a^2 \frac{p}{h}(\pi + \frac{2}{h})}}, \frac{1}{q + \sqrt{\xi + a^2 \frac{p}{h}(1 + \frac{1}{h})}} \right\}$$

$$= \frac{1}{q + \sqrt{\xi + a^2 \frac{p}{h}(\pi + \frac{2}{h})}},$$

we have

$$\|B(\varepsilon)(\lambda - T_0)^{-1}\| < 1 \quad \text{for } |Im\lambda| \geq h \ \text{ or } \ Re\lambda \leq -h.$$

Hence, $I - B(\varepsilon)(\lambda - T_0)^{-1}$ is invertible with bounded inverse outside S_h. Then, Eq. (3.15) implies that $\lambda - T(\varepsilon)$ is invertible with bounded inverse and we obtain

$$(\lambda - T(\varepsilon))^{-1} = (\lambda - T_0)^{-1}[I - B(\varepsilon)(\lambda - T_0)^{-1}]^{-1}. \tag{3.18}$$

Consequently $\lambda \in \rho(T(\varepsilon))$. So, the spectrum of the operator $T(\varepsilon)$ lies in the half-strip S_h. $\qquad\square$

These results are of importance to prove the aim of this subsection.

Theorem 3.3. *Assume that hypotheses (H1)-(H3) hold. Then, for small enough $|\varepsilon|$, the system of root vectors of the operator $T(\varepsilon)$ forms a basis with parentheses in $\mathcal{H}$.*$\diamond$

Remark 3.2. (*i*) Theorem 3.3 guarantees basicity with parentheses not only for the sum of two operators such as in [24, *Theorem* 2] but for an analytic operator. Further, we prove that even if Eq. (1.2) considered in [24] is not verified, we can get a similar result.
(*ii*) Theorem 3.3 improves [13, *Theorem* 3.4] since we prove the existence of a basis with parentheses of root vectors of $T(\varepsilon)$ where the eigenvalues of T_0 are with finite multiplicity instead of multiplicity one. Indeed, in order to prove the existence of a Riesz basis related to the eigenvectors of $T(\varepsilon)$, the authors in [13] used a spectral analysis method based on the fact that the eigenvalues of T_0 are with multiplicity one. However, this spectral analysis can not be applied when the eigenvalues of T_0 are with finite multiplicity. $\qquad\diamond$

Before going further, we recall the following result stated in [9].

Theorem 3.4. [*9, Theorem 4.3.2*] *Suppose that hypotheses (H1)-(H3) are satisfied. Then, for small enough $|\varepsilon|$, the spectrum of the operator $T(\varepsilon)$ is constituted by isolated eigenvalues satisfying*

$$n(r, T(\varepsilon)) = n(r, T_0) + O(1) \quad i.e., \quad |n(r, T(\varepsilon)) - n(r, T_0)| < C_3 \quad as \ \ r \ \to \infty,$$

where $n(r, T_0)$ (respectively, $n(r, T(\varepsilon))$) denotes the sum of multiplicities of all eigenvalues of T_0 (respectively, $T(\varepsilon)$) contained in the disk $\{\lambda \in \mathbb{C} \ \ such \ that \ \ |\lambda| < r\}$ and C_3 is a constant. $\qquad\diamond$

Proof of Theorem 3.3.

Let $\lambda \in \mathbb{C}$. In view of Proposition 3.3, the spectrum of $T(\varepsilon)$ lies in the half-strip $S_h := \{\lambda \text{ such that } |Im\lambda| < h \text{ and } Re\lambda > -h\}$, for $|\varepsilon| < \frac{1}{q+\sqrt{\xi+a^2\frac{p}{h}(\pi+\frac{2}{h})}}$. So, let $(\Delta_k)_{k\geq 1}$ be the rectangles bounded by the straight lines $Im\lambda = \pm h$, $Re\lambda = r_k$ and $Re\lambda = r_{k-1}$, where $r_0 = -h$ and $r_k \to \infty$ (see Figure 2).

We note here that the numbers r_k are chosen in such away that the boundary $\partial\Delta_k$ of any rectangle Δ_k does not pass through the eigenvalues of the operator $T(\varepsilon)$.
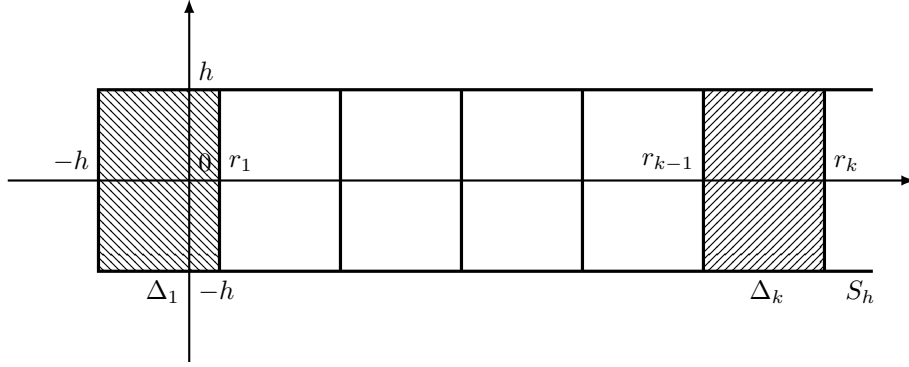

Figure 2

Then, for $|\varepsilon| < \frac{1}{q+\sqrt{\xi+a^2\frac{p}{h}(\pi+\frac{2}{h})}}$ we have

$$\sum_{k=1}^{n} \frac{-1}{2\pi i} \int_{\partial\Delta_k} (\lambda - T(\varepsilon))^{-1}d\lambda = \sum_{k=1}^{n} P_k(\varepsilon),$$

where $P_k(\varepsilon)$ designates the spectral projection corresponding to the spectrum of $T(\varepsilon)$ inside Δ_k.

To prove our result, it suffices to show that

$$\sup_n \left| \sum_{k=1}^{n} \frac{-1}{2\pi i} \int_{\partial\Delta_k} (\lambda - T(\varepsilon))^{-1}d\lambda \right| < \infty. \qquad (3.19)$$

In order to do, so we are going first to estimate $\|(\lambda - T(\varepsilon))^{-1}\|$ for:

(i) $|Im\lambda| = \tau \geq h$ and $|\varepsilon| < \dfrac{1}{q + \sqrt{\xi + a^2\frac{p}{h}(\pi + \frac{2}{h})}}$.

(ii) $Re\lambda = -\tau$ and $|\varepsilon| < \dfrac{1}{q + \sqrt{\xi + a^2\frac{p}{h}(1 + \frac{1}{h})}}$.

For this purpose, let us consider $\lambda = \sigma + i\tau$.

(i) In view of Eq. (3.16), we have

$$\|B(\varepsilon)(\lambda - T_0)^{-1}\| < \sqrt{N(\varepsilon, a, p, q, \tau)} < 1, \tag{3.20}$$

where

$$N(\varepsilon, a, p, q, \tau) := \frac{|\varepsilon|^2}{(1 - |\varepsilon|q)^2} \left(\xi + a^2 \frac{p}{\tau} \left(\pi + \frac{2}{\tau} \right) \right).$$

Further,

$$\|(\lambda - T_0)\|^{-1} \leq \frac{1}{|Im\lambda|} = \frac{1}{\tau}. \tag{3.21}$$

Then, Eqs (3.20) and (3.21) yield

$$
\begin{aligned}
\left\|(\lambda - T(\varepsilon))^{-1}\right\| &= \left\|(\lambda - T_0)^{-1} \left[I - B(\varepsilon)(\lambda - T_0)^{-1} \right]^{-1}\right\| \\
&\leq \left\|(\lambda - T_0)^{-1}\right\| \left\|\left[I - B(\varepsilon)(\lambda - T_0)^{-1} \right]^{-1}\right\| \\
&\leq \frac{1}{\tau} \left(1 - \sqrt{N(\varepsilon, a, p, q, \tau)} \right)^{-1}.
\end{aligned}
$$

(ii) Eq. (3.17) implies that

$$\|B(\varepsilon)(\lambda - T_0)^{-1}\| < \sqrt{N_1(\varepsilon, a, p, q, \tau)} < 1, \tag{3.22}$$

where

$$N_1(\varepsilon, a, p, q, \tau) := \frac{|\varepsilon|^2}{(1 - |\varepsilon|q)^2} \left(\xi + a^2 \frac{p}{\tau} \left(1 + \frac{1}{\tau} \right) \right).$$

Furthermore, since

$$
\begin{aligned}
\|(\lambda - T_0)^{-1}\| &\leq \frac{1}{d(\lambda, \sigma(T_0))} \\
&\leq \frac{1}{|Re\lambda - \lambda_n|}, \qquad \lambda_n \in \sigma(T_0),
\end{aligned}
$$

then we get

$$
\begin{aligned}
\|(\lambda - T_0)^{-1}\| &\leq \frac{1}{|-\tau - \lambda_n|}, \qquad Re\lambda = -\tau \\
&< \frac{1}{\tau}. \tag{3.23}
\end{aligned}
$$

Consequently, due to Eqs (3.22) and (3.23) we obtain

$$
\begin{aligned}
\left\|(\lambda - T(\varepsilon))^{-1}\right\| &= \left\|(\lambda - T_0)^{-1} \left[I - B(\varepsilon)(\lambda - T_0)^{-1} \right]^{-1}\right\| \\
&\leq \frac{1}{\tau} \left(1 - \sqrt{N_1(\varepsilon, a, p, q, \tau)} \right)^{-1}.
\end{aligned}
$$

Now, to prove Eq. (3.19) it remains to show the existence of vertical segments in the half-strip S_h that tend to infinity and on which $(\lambda - T(\varepsilon))^{-1}$ is uniformly bounded (see [24, p. 292]).

Let us begin with the boundedness of $\|(\lambda - T(\varepsilon))^{-1}\|$. Let $f, g \in \mathcal{H}$ and consider the scalar function $F_\varepsilon(\lambda)$ defined as

$$F_\varepsilon(\lambda) = \langle (\lambda - T(\varepsilon))^{-1} f, g \rangle.$$

It is easy to see that

$*$ $F_\varepsilon(\lambda)$ is meromorphic and belongs to the Carleman-class $\mathcal{C}_P, P > 1$. In fact, due to [22, p. 13] the set of the Carleman-class $\mathcal{C}_P$ is a two-sided ideal of the algebra of bounded operators $L(H)$. Further, the resolvent of T_0 belongs to the Carleman-class $\mathcal{C}_P$ (see the proof of Theorem 3.2). Then, in view of Eq. (3.18) the resolvent of $T(\varepsilon)$ belongs to the Carleman-class $\mathcal{C}_P$.

$*$ The poles of $F_\varepsilon(\lambda)$ lie in the strip $|Im\lambda| < h$. Indeed, in view of [21, p. 38], the poles of $(\lambda - T(\varepsilon))^{-1}$ are exactly the eigenvalues of $T(\varepsilon)$ which lies in the half-strip S_h.

$*$ $|F_\varepsilon(\lambda)| \leq \dfrac{1}{\tau} \left(1 - \sqrt{N(\varepsilon, a, p, q, \tau)} \right)^{-1}$, for $|Im\lambda| = \tau = h + \delta$, $\delta > 0$ and $|\varepsilon| < \dfrac{1}{q + \sqrt{\xi + a^2 \frac{p}{h}(\pi + \frac{2}{h})}}$.

Then, in view of Lemma 2.1 we have

$$\ln |F_\varepsilon(\lambda)| \quad \leq \quad C' \left(\frac{1}{\tau} \left(1 - \sqrt{N(\varepsilon, a, p, q, \tau)} \right)^{-1} + \sup_{|t-r| \leq r^\eta} \left(n(t+1, F_\varepsilon) - n(t, F_\varepsilon) \right) \right),$$

for $|Im\lambda| \leq \tau$ and $Re\lambda = r_n \to \infty$ outside an exceptional set of disks $\mathcal{D}$, with $r = |\lambda|$. On the other hand, in virtue of Theorem 3.4 there exists a positive constant W such that for $|\varepsilon| < W$ we have

$$n(r, T(\varepsilon)) = n(r, T_0) + O(1). \tag{3.24}$$

Hence, hypothesis $(H2)$ and Eq. (3.24) imply that for $|\varepsilon| < W$

$$
\begin{aligned}
n(t+1, F_\varepsilon) - n(t, F_\varepsilon) \quad &= n(t+1, T(\varepsilon)) - n(t, T(\varepsilon)) \\[2mm]
&= [n(t+1, T_0) + O(1)] - [n(t, T_0) + O(1)] \\[2mm]
&= O(1) + [n(t+1, T_0) - n(t, T_0)] \\[2mm]
&\leq O(1) + p = p'.
\end{aligned}
$$

Consequently, for $|\varepsilon| < V := \min \left\{ W, \dfrac{1}{q + \sqrt{\xi + a^2 \frac{p}{h}(\pi + \frac{2}{h})}} \right\}$ we have

$$|F_\varepsilon(\lambda)| \leq C_\varepsilon,$$

where C_ε is a constant independent of f, g.

Therefore, for $|\varepsilon| < V$ we obtain

$$\|(\lambda - T(\varepsilon))^{-1}\| \leq C_\varepsilon,$$

where $|Im\lambda| \leq \tau$ and $Re\lambda = r_n \to \infty$ outside an exceptional set of disks $\mathcal{D}$. Then, for $|\varepsilon| < V$ we have

$$\left| \sum_{k=1}^{n} \frac{-1}{2\pi i} \int_{\partial \Delta_k} (\lambda - T(\varepsilon))^{-1} \, d\lambda \right| < \infty.$$

Hence,

$$\sup_n \left| \sum_{k=1}^{n} \frac{-1}{2\pi i} \int_{\partial \Delta_k} (\lambda - T(\varepsilon))^{-1} \, d\lambda \right| < \infty.$$

Thus,

$$\sup_n \left\| \sum_{k=1}^{n} P_k(\varepsilon) \right\| < \infty.$$

As a consequence, due to Theorem 2.2, we claim that the family $(R(P_k(\varepsilon)))_{k \geq 1}$ forms a basis in $\mathcal{H}$ which means that the family of root vectors of $T(\varepsilon)$ forms a basis with parentheses in $\mathcal{H}$.

To complete the proof of our result, we show by a similar way as [24] the existence of vertical segments that do not pass through the eigenvalues of the operator $T(\varepsilon)$. Indeed, in each rectangle bounded by the straight lines $Re\lambda = n$, $Re\lambda = n+1$ and $Im\lambda = \pm h$, there are at most p' points of the eigenvalues $\lambda_k(\varepsilon)$ for $|\varepsilon| < W$. Hence, the projection of the disks from $\mathcal{D}$ onto the real axis does not fill the interval $[n, n+1]$. In fact, it suffices to choose $d < \frac{1}{2p'}$ (where d is the total radii of the disks from $\mathcal{D}$ inside each rectangle). So, there exists a vertical segment in this rectangle that does not intersect $\mathcal{D}$ (see Figure 3). Moreover, the vertical segments can be chosen in such a way that only points $\lambda_k(\varepsilon)$ with $|Re(\lambda_k(\varepsilon)) - Re(\lambda_j(\varepsilon))| < d$ fall between the neighboring segments.
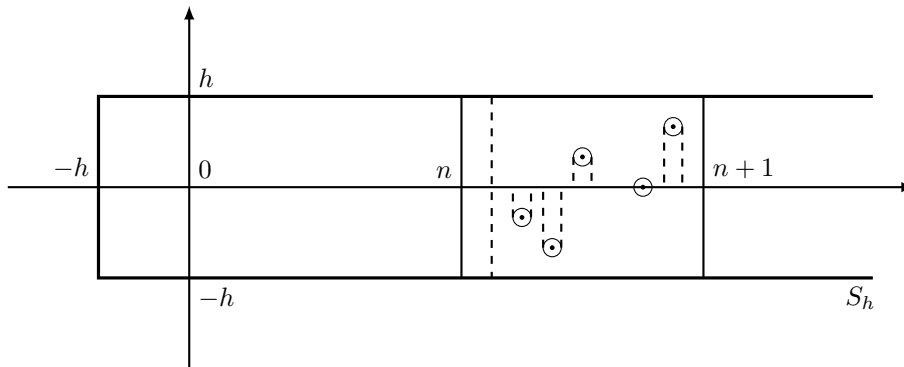

Figure 3

Corollary 3.2. *Assume that hypotheses* $(H1)$ *and* $(H3)$ *and Eq.* (3.2) *hold. Then, for* $\beta \in]0, 1 + \frac{\alpha-1}{2}[$ *and small enough* $|\varepsilon|$, *the system of root vectors of the operator* $T(\varepsilon)$ *forms a basis with parentheses in* $\mathcal{H}$. ◇

Proof. Using $(H3)$ and making the same reasoning as the one developed in the proof of Lemma 3.1, we get for $|\varepsilon| < \frac{1}{q}$

$$\frac{\|B(\varepsilon)\varphi_n\|^2}{|\lambda - \lambda_n|^2} \leq \frac{|\varepsilon|^2}{(1 - |\varepsilon|q)^2} \left(\frac{a^2}{|\lambda - \lambda_n|^2} + \frac{2ab\lambda_n^\beta}{|\lambda - \lambda_n|^2} + \frac{b^2\lambda_n^{2\beta}}{|\lambda - \lambda_n|^2} \right).$$

Now, let $\sigma = Re\lambda$. Then there exists $k \in \mathbb{N}^*$ such that $\lambda_{k-1} \leq \sigma$ and $\lambda_k > \sigma$. Since $\lambda_n^{1-\alpha} \geq \frac{n-1}{p} + \lambda_1^{1-\alpha}$, then $\lambda_n \geq \left(\frac{n-1}{p} + \lambda_1^{1-\alpha}\right)^{\frac{1}{1-\alpha}}$. Hence, for $n < k - 1$ we obtain

$$
\begin{aligned}
\sigma - \lambda_n &> \lambda_{k-1} - \lambda_n \\
&\geq \lambda_n^\beta(\lambda_{k-1}^{1-\beta} - \lambda_n^{1-\beta}) \\
&\geq \lambda_n^\beta \left(\left(\frac{k-2}{p} + \lambda_1^{1-\alpha}\right)^{\frac{1-\beta}{1-\alpha}} - \left(\frac{n-1}{p} + \lambda_1^{1-\alpha}\right)^{\frac{1-\beta}{1-\alpha}} \right).
\end{aligned}
\tag{3.25}
$$

Equivalently to Eq. (3.25), for $n > k$ we have

$$\lambda_n - \sigma > \lambda_n^\beta \left(\left(\frac{n-1}{p} + \lambda_1^{1-\alpha}\right)^{\frac{1-\beta}{1-\alpha}} - \left(\frac{k-1}{p} + \lambda_1^{1-\alpha}\right)^{\frac{1-\beta}{1-\alpha}} \right). \tag{3.26}$$

Two cases are presented: If $\beta \in]0, \alpha]$, then we have $\frac{1-\beta}{1-\alpha} \geq 1$. Hence, Eq. (3.25) implies that

$$\sigma - \lambda_n > \lambda_n^\beta \frac{(k-1-n)^{\frac{1-\beta}{1-\alpha}}}{p^{\frac{1-\beta}{1-\alpha}}} \tag{3.27}$$

and Eq. (3.26) yields

$$\lambda_n - \sigma > \lambda_n^\beta \frac{(n-k)^{\frac{1-\beta}{1-\alpha}}}{p^{\frac{1-\beta}{1-\alpha}}}. \tag{3.28}$$

Consequently, it follows from Eqs (3.9), (3.27) and (3.28) that

$$
\begin{aligned}
\sum_{n=1}^\infty \frac{\lambda_n^{2\beta}}{|\lambda_n - \sigma|^2} &< \frac{2}{C^2} + p^{\frac{2(1-\beta)}{1-\alpha}} \left(\sum_{n<k-1} \frac{1}{(k-1-n)^{\frac{2(1-\beta)}{1-\alpha}}} + \sum_{n>k}^\infty \frac{1}{(n-k)^{\frac{2(1-\beta)}{1-\alpha}}} \right) \\
&\leq \frac{2}{C^2} + p^{\frac{2(1-\beta)}{1-\alpha}} \left(\sum_{n<k-1} \frac{1}{(k-1-n)^{\frac{2(1-\beta)}{1-\alpha}}} + \sum_{m=1}^\infty \frac{1}{m^{\frac{2(1-\beta)}{1-\alpha}}} \right) \\
&< \frac{2}{C^2} + 2p^{\frac{2(1-\beta)}{1-\alpha}} \sum_{m=1}^\infty \frac{1}{m^{\frac{2(1-\beta)}{1-\alpha}}} =: \xi_1' < \infty.
\end{aligned}
\tag{3.29}
$$

Now, if $\beta \in]\alpha, 1 + \frac{\alpha-1}{2}[$. Then, we have $0 < \frac{-\alpha+\beta}{1-\alpha} < \frac{1}{2}$ and $\frac{1}{2} < 1 - \frac{-\alpha+\beta}{1-\alpha} = \frac{1-\beta}{1-\alpha} < 1$. So, in view of [22, *p.* 33] and Eq. (3.25) we get

$$\sigma - \lambda_n > \gamma' \lambda_n^\beta \frac{\left(\frac{1-\beta}{1-\alpha}\right)(k-1-n)^{\frac{1-\beta}{1-\alpha}}}{p^{\frac{1-\beta}{1-\alpha}}}, \quad 0 < \gamma' < 1. \tag{3.30}$$

Further, based on [22, *p.* 33] and Eq. (3.26) we obtain

$$\lambda_n - \sigma > \gamma'' \lambda_n^\beta \frac{\left(\frac{1-\beta}{1-\alpha}\right)(n-k)^{\frac{1-\beta}{1-\alpha}}}{p^{\frac{1-\beta}{1-\alpha}}}, \quad 0 < \gamma'' < 1. \tag{3.31}$$

Hence, similarly to Eq. (3.29), Eqs (3.9), (3.30) and (3.31) imply that

$$\sum_{n=1}^\infty \frac{\lambda_n^{2\beta}}{|\lambda_n - \sigma|^2} < \frac{2}{C^2} + \frac{2p^{\frac{2(1-\beta)}{1-\alpha}}}{\gamma_1'^2 \left(\frac{1-\beta}{1-\alpha}\right)^2} \sum_{m=1}^\infty \frac{1}{m^{\frac{2(1-\beta)}{1-\alpha}}} =: \xi_2' < \infty,$$

where $\gamma_1' := \min\{\gamma', \gamma''\}$. Consequently, for $\beta \in]0, 1 + \frac{\alpha-1}{2}[$ we get

$$\sum_{n=1}^\infty \frac{\lambda_n^{2\beta}}{|\lambda - \lambda_n|^2} < \sum_{n=1}^\infty \frac{\lambda_n^{2\beta}}{|\sigma - \lambda_n|^2} < \max\{\xi_1', \xi_2'\} =: \xi_2'.$$

On the other hand, if we replace β by $\frac{\beta}{2}$ we get

$$\sum_{n=1}^\infty \frac{\lambda_n^\beta}{|\lambda - \lambda_n|^2} < \frac{2}{C^2} + \frac{2p^{\frac{2-\beta}{1-\alpha}}}{\gamma_1'^2 \left(\frac{1-\frac{\beta}{2}}{1-\alpha}\right)^2} \sum_{m=1}^\infty \frac{1}{m^{\frac{2-\beta}{1-\alpha}}} =: \xi_3' < \infty.$$

Hence, the series $\sum_n \frac{2ab\lambda_n^\beta}{|\sigma-\lambda_n|^2}$ and $\sum_n \frac{b^2\lambda_n^{2\beta}}{|\lambda-\lambda_n|^2}$ are convergent. So, let ξ' be a positive constant satisfying

$$\sum_{n=1}^\infty \frac{2ab\lambda_n^\beta}{|\lambda - \lambda_n|^2} + \sum_{n=1}^\infty \frac{b^2\lambda_n^{2\beta}}{|\lambda - \lambda_n|^2} < b^2\xi_2' + 2ab\xi_3' =: \xi'.$$

Furthermore, it follows from [9, Corollary 3.2] that for small enough $|\varepsilon|$ and $\beta \in]0, 1 + \frac{\alpha-1}{2}[$ we have

$$n(r, T(\varepsilon)) = n(r, T_0) + O(1).$$

To get the desired result, we advise that the rest of the proof is similar to that of Theorem 3.3. $\qquad\square$

4. Applications

4.1. Application to a Gribov operator in Bargmann space

We are interested in a family of non self-adjoint operators, said of Gribov, studied by the specialists of physics of height energy. A representant of this family is

a combination between the creation operator A^* and the annihilation operator A ([1], [2] and [15]) given by:

$$(A^*A)^3 + \varepsilon A^*(A + A^*)A + \varepsilon^2(A^*A)^{3u_2} + ... + \varepsilon^k(A^*A)^{3u_k} + ...,$$

where $\varepsilon \in \mathbb{C}$ and $(u_k)_{k \in \mathbb{N}}$ is a strictly decreasing sequence with strictly positive terms such that $u_0 = 1$ and $u_1 = \frac{1}{2}$.

We define the Bargmann space $\mathcal{B}$ by:

$$\mathcal{B} = \left\{ \varphi : \mathbb{C} \longrightarrow \mathbb{C} \text{ entire such that } \int_{\mathbb{C}} e^{-|z|^2} |\varphi(z)|^2 dz d\bar{z} < \infty \right\}.$$

This space is equipped with the following scalar product:

$$\begin{cases} \langle ., . \rangle : \mathcal{B} \times \mathcal{B} \longrightarrow \mathbb{C} \\ \qquad (\varphi, \psi) \longrightarrow \langle \varphi, \psi \rangle = \int_{\mathbb{C}} e^{-|z|^2} \varphi(z) \bar{\psi}(z) dz d\bar{z} \end{cases}$$

and its associated norm is denoted by $\|.\|$.

The expressions of the operators A and A^* are given by:

$$\begin{cases} A : \mathcal{D}(A) \subset \mathcal{B} \longrightarrow \mathcal{B} \\ \qquad \varphi \longrightarrow A\varphi(z) = \dfrac{d\varphi}{dz}(z) \\ \mathcal{D}(A) = \{\varphi \in \mathcal{B} \text{ such that } A\varphi \in \mathcal{B}\} \end{cases}$$

and

$$\begin{cases} A^* : \mathcal{D}(A^*) \subset \mathcal{B} \longrightarrow \mathcal{B} \\ \qquad \varphi \longrightarrow A^*\varphi(z) = z\varphi(z) \\ \mathcal{D}(A^*) = \{\varphi \in \mathcal{B} \text{ such that } A^*\varphi \in \mathcal{B}\}. \end{cases}$$

We consider the problem on $E = \{\varphi \in \mathcal{B} \text{ such that } \varphi(0) = 0\}$ and we denote by T_0 and H_1 the following operators:

$$\begin{cases} T_0 : \mathcal{D}(T_0) \subset E \longrightarrow E \\ \qquad \varphi \longrightarrow T_0\varphi(z) = (A^*A)^3\varphi(z) \\ \mathcal{D}(T_0) = \{\varphi \in E \text{ such that } T_0\varphi \in E\}, \end{cases}$$

and

$$\begin{cases} H_1 : \mathcal{D}(H_1) \subset E \longrightarrow E \\ \qquad \varphi \longrightarrow H_1\varphi(z) = A^*(A + A^*)A\varphi(z) \\ \mathcal{D}(H_1) = \{\varphi \in E \text{ such that } H_1\varphi \in E\}. \end{cases}$$

Now, we recall a straightforward, but useful result from [12].

Proposition 4.1. *[12, Proposition 6.2] We have the following assertions:*

(i) T_0 *is a self-adjoint operator.*
(ii) *The resolvent set of* T_0 *is compact.*
(iii) $\{e_n(z) = \frac{z^n}{\sqrt{n!}}\}_1^\infty$ *is a system of eigenvectors associated to the eigenvalues* $\{n^3\}_{n\geq 1}$ *of* T_0. $\diamondsuit$

Proposition 4.2. *The resolvent of the operator* T_0 *belongs to the Carleman-class* $\mathcal{C}_P$ *for any* $P > \frac{1}{3}$. $\diamondsuit$

Due to Proposition 4.1, T_0 is a self-adjoint operator with compact resolvent in E. Then, let

$$T_0 = \sum_{n=1}^\infty n^3 \langle ., e_n \rangle e_n$$

be its spectral decomposition. So, for a strictly decreasing sequence $(u_k)_{k\in\mathbb{N}}$ with strictly positive terms such that $u_0 = 1$ and $u_1 = \frac{1}{2}$, the operators $(T_0^{u_k})_{k\geq 0}$ are defined by:

$$\begin{cases} T_0^{u_k} : \mathcal{D}(T_0^{u_k}) \subset E \longrightarrow E \\ \qquad \varphi \longrightarrow T_0^{u_k}\varphi = \sum_{n=1}^\infty n^{3u_k} \langle \varphi, e_n \rangle e_n \\ \mathcal{D}(T_0^{u_k}) = \{\varphi \in E \text{ such that } \sum_{n=1}^\infty n^{6u_k} |\langle \varphi, e_n \rangle|^2 < \infty\}. \end{cases}$$

It is easy to check that for all $k \geq 0$, $\mathcal{D}(T_0^{u_k}) \subset \mathcal{D}(T_0^{u_{k+1}})$. Then, $\bigcap_{k\geq 2} \mathcal{D}(T_0^{u_k}) = \mathcal{D}(T_0^{u_2})$.
Let $\mathcal{D} = \mathcal{D}(T_0^{u_2}) \cap \mathcal{D}(H_1)$, T_1, $(T_k)_{k\geq 2}$ be the restrictions of H_1 and $T_0^{u_k}$ to $\mathcal{D}$, respectively. So, the operators $(T_k)_{k\geq 1}$ have the same domain $\mathcal{D}$ and we have $\mathcal{D}(T_0) \subset \mathcal{D}$.

Proposition 4.3. *[12, Proposition 6.3] There exist positive constants* $a, b, q > 0$ *and* $\beta \in \left[\frac{1}{2}, 1\right]$ *such that for all* $\varphi \in \mathcal{D}(T_0)$ *and for all* $k \geq 1$ *we have*

$$\|T_k \varphi\| \leq q^{k-1}(a\|\varphi\| + b\|T_0\varphi\|^\beta \|\varphi\|^{1-\beta}). \qquad \diamondsuit$$

Remark 4.1. In Proposition 4.3, we take $q = 1$ and $a = b = 1 + 2\sqrt{2}$. $\diamondsuit$

Proposition 4.4. *For* $|\varepsilon| < 1$, *the series* $\sum_{k\geq 0} \varepsilon^k T_k \varphi$ *converges for all* $\varphi \in \mathcal{D}(T_0)$. *If we denote its sum by* $T(\varepsilon)\varphi$, *then we define a linear operator* $T(\varepsilon)$ *with domain* $\mathcal{D}(T_0)$. *Also, for* $|\varepsilon| < \frac{1}{1+\beta a}$, *the operator* $T(\varepsilon)$ *is closed.* $\diamondsuit$

The main results of this part are formulated as follows:

Proposition 4.5. *For* $|\varepsilon| < \frac{1}{1+\beta a}$ *and* $\beta \in [\frac{1}{2}, \frac{5}{6}[$, *the system of root vectors of the operator* $T(\varepsilon)$ *is complete in* E. $\diamondsuit$

Proof. Let λ_n be the eigenvalue number n of $(A^*A)^3$. It is easy to see that

$$\lambda_{n+p}^{\frac{1}{3}} - \lambda_n^{\frac{1}{3}} = (n+p) - n = p \geq 1, \quad (\text{where } \alpha = \frac{2}{3}). \tag{4.1}$$

Consequently, Corollary 3.1, Propositions 4.1, 4.2 and 4.4 and Eq. (4.1) imply that the system of root vectors of the operator $T(\varepsilon)$ is complete in E for $|\varepsilon| < \frac{1}{1+\beta a}$.   □

We have proved that the system of root vectors of the operator $T(\varepsilon)$ is complete in E. Now, it remains to show that it forms a basis with parentheses in E.

Theorem 4.1. *For small enough $|\varepsilon|$ and $\beta \in [\frac{1}{2}, \frac{5}{6}[$, the system of root vectors of the Gribov operator forms a basis with parentheses in E.*   ◇

Proof. It suffices to apply Corollary 3.2, Propositions 4.1 and 4.4 and Eq. (4.1).   □

Remark 4.2. Theorem 4.1 ameliorates Theorem 4.1 stated in [4]. In fact, we have proved that for $\beta \in [\frac{1}{2}, \frac{5}{6}[$ the system of root vectors of the Gribov operator forms a basis with parentheses in E; while in [4], the authors showed the existence of a Riesz basis of finite-dimensional invariant subspaces for $\beta = \frac{2}{3}$.   ◇

4.2. Application to a problem of radiation of a vibrating structure in a light fluid

An elastic membrane is stimulated by a harmonic force $F(x)e^{-i\omega t}$. It occupies the domain $-L < x < L$ of the plane $z = 0$. The two half-spaces $z < 0$ and $z > 0$ are filled with gas. The mechanical parameters of the membrane are E the Young modulus, ν the Poisson ratio, m the surface density, h the thickness of the membrane and $D := \frac{Eh^3}{12(1-\nu^2)}$ the rigidity. The fluid is characterized by ρ_0 the density, c the sound speed and $k := \frac{\omega}{c}$ the wave number.

Now, let us consider the following boundary value problem:

$$\left(\frac{d^4}{dx^4} - \frac{m\omega^2}{D} \right) u(x)$$

$$-i\rho_0 \int_{-L}^{L} H_0(k|x-x'|) \left(\frac{\omega^2}{D} - \frac{1}{m} \left(\frac{d^4}{dx^4} - \left(\frac{d^4}{dx^4} \right)^{\frac{1}{2}} \right) \right) u(x')dx' = \frac{F(x)}{D}, \quad (4.2)$$

for all $x \in]-L, L[$ where u denotes the displacement of the membrane such that $u(x) = \dfrac{\partial u(x)}{\partial x} = 0$ for $x = -L$ and $x = L$ and H_0 is the Hankel function of the first kind and order 0 (see [20, p. 11]).

The problem (4.2) satisfy the following system:

$$\left(\frac{d^4}{dx^4} - \frac{m\omega^2}{D} \right) u(x) = \frac{1}{D}(F(x) - P(x)) \quad \text{for all} \quad x \in]-L, L[,$$

where

$$u(x) = \frac{\partial u(x)}{\partial x} = 0 \quad \text{for} \quad x = -L \quad \text{and} \quad x = L,$$

$$P(x) = \lim_{\eta \to 0^+} (p(x, \eta) - p(x, -\eta))$$

and

$$p(x, z)$$

$$= -sgn\ zi\frac{\rho_0}{2} \int_{-L}^{L} H_0(k\sqrt{(x - x')^2 + z^2}) \left(\omega^2 - \frac{D}{m} \left(\frac{d^4}{dx^4} - \left(\frac{d^4}{dx^4} \right)^{\frac{1}{2}} \right) \right) u(x')dx',$$

for $z < 0$ or $z > 0$ such that p designates the acoustic pressure in the fluid.

In order to study this problem, we consider the following operators:

$$\begin{cases} T_0 : \mathcal{D}(T_0) \subset L^2(]-L, L[) \longrightarrow L^2(]-L, L[) \\ \\ \varphi \longrightarrow T_0\varphi(x) = \dfrac{d^4\varphi}{dx^4} \\ \\ \mathcal{D}(T_0) = H_0^2(]-L, L[) \cap H^4(]-L, L[) \end{cases}$$

and

$$\begin{cases} K : L^2(]-L, L[) \longrightarrow L^2(]-L, L[) \\ \\ \varphi \longrightarrow K\varphi(x) = \dfrac{i}{2} \int_{-L}^{L} H_0(k|x - x'|)\varphi(x')dx'. \end{cases}$$

Now, we recall the following result from [20].

Lemma 4.1. [*20, Lemmas 3.1 and 3.2 and Theorem 3.1*] *The following assertions hold:*
(i) T_0 is a self-adjoint operator.
(ii) The injection from $\mathcal{D}(T_0)$ into $L^2(]-L, L[)$ is compact.
(iii) The spectrum of T_0 is constituted only of point spectrums which are positive, denumerable and of which the multiplicity is one and which have no finite limit points and satisfies

$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n \to +\infty.$$

Further,

$$\left(\frac{(2n + 1)\pi}{4L} \right)^4 \le \lambda_n \le \left(\frac{(2n + 3)\pi}{4L} \right)^4, \quad i.e., \quad \lambda_n \sim_{+\infty} \left(\frac{n\pi}{2L} \right)^4.$$

(iv) The resolvent of the operator T_0 belongs to the Carleman-class $\mathcal{C}_P$ for any $P > \frac{1}{4}$.
$\diamondsuit$

Due to Lemma 4.1, T_0 is a self-adjoint operator and has a compact resolvent. Then, let

$$T_0\varphi = \sum_{n=1}^{\infty} \lambda_n \langle \varphi, \varphi_n \rangle \varphi_n$$

be its spectral decomposition, where $\lambda_n = \kappa n^4$ is the nth eigenvalue of T_0 associated to the eigenvector $\varphi_n(x) = \mu e^{\sqrt[4]{\lambda_n}x} + \eta e^{-\sqrt[4]{\lambda_n}x} + \theta e^{i\sqrt[4]{\lambda_n}x} + \delta e^{-i\sqrt[4]{\lambda_n}x}$ (see [20, p. 7]). Hence, we define the operator B by:

$$\begin{cases} B = T_0^{\frac{1}{2}} : \mathcal{D}(B) \subset L^2\big(]-L, L[\big) \longrightarrow L^2\big(]-L, L[\big) \\ \qquad\qquad\qquad\qquad \varphi \longrightarrow B\varphi(x) = \left(\dfrac{d^4\varphi}{dx^4}\right)^{\frac{1}{2}} \\ \mathcal{D}(B) = \left\{ \varphi \in L^2(]-L, L[) \text{ such that } \displaystyle\sum_{n=1}^{\infty} \lambda_n |\langle \varphi, \varphi_n \rangle|^2 < \infty \right\} \end{cases}$$

and we consider the following eigenvalue problem:
Find the values $\lambda(\varepsilon) \in \mathbb{C}$ for which there is a solution $\varphi \in H_0^2\big(]-L, L[\big) \cap H^4\big(]-L, L[\big)$, $\varphi \neq 0$ for the equation

$$T_0\varphi + \varepsilon K(T_0 - B)\varphi = \lambda(\varepsilon)(I + \varepsilon K)\varphi \tag{4.3}$$

where $\lambda = \frac{m\omega^2}{D}$ and $\varepsilon = \frac{2\rho_0}{m}$.
Note that both λ and φ depend on the value of ε. So, we denote this by $\lambda := \lambda(\varepsilon)$ and $\varphi := \varphi(\varepsilon)$.

For $|\varepsilon| < \frac{1}{\|K\|}$, the operator $I + \varepsilon K$ is invertible. Then, the problem (4.3) becomes:
Find the values $\lambda(\varepsilon) \in \mathbb{C}$ for which there is a solution $\varphi \in H_0^2\big(]-L, L[\big) \cap H^4\big(]-L, L[\big)$, $\varphi \neq 0$ for the equation

$$(I + \varepsilon K)^{-1}T_0\varphi + \varepsilon(I + \varepsilon K)^{-1}K(T_0 - B)\varphi = \lambda(\varepsilon)\varphi. \tag{4.4}$$

The problem (4.4) is equivalent to:
Find the values $\lambda(\varepsilon) \in \mathbb{C}$ for which there is a solution $\varphi \in H_0^2\big(]-L, L[\big) \cap H^4\big(]-L, L[\big)$, $\varphi \neq 0$ for the equation

$$\big(T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \ldots + \varepsilon^n T_n + \ldots\big)\varphi = \lambda(\varepsilon)\varphi,$$

where $T_n := (-1)^n K^n \left(\dfrac{d^4}{dx^4}\right)^{\frac{1}{2}}$, for all $n \geq 1$.

Proposition 4.6. *[11, Proposition 4.1] The following properties hold:*
(i) There exist positive constants $a, b, q > 0$ and $\beta \in \left[\frac{1}{2}, 1\right]$ such that for all $\varphi \in \mathcal{D}(T_0)$ and for all $k \geq 1$ we have

$$\|T_k\varphi\| \leq q^{k-1}(a\|\varphi\| + b\|T_0\varphi\|^{\beta}\|\varphi\|^{1-\beta}).$$

Note that it suffices to take $a = b = q = \|K\|$.
(ii) For $|\varepsilon| < \frac{1}{\|K\|}$, the series $\sum_{k \geq 0} \varepsilon^k T_k\varphi$ converges for all $\varphi \in \mathcal{D}(T_0)$. If we denote its sum by $T(\varepsilon)\varphi$, we define a linear operator $T(\varepsilon)$ with domain $\mathcal{D}(T_0)$. For $|\varepsilon| < \frac{1}{\|K\|(1+\beta)}$, the operator $T(\varepsilon)$ is closed. ◇

Using the results described above, we can now prove the objective of this part.

Proposition 4.7. *For $|\varepsilon| < \frac{1}{\|K\|(1+\beta)}$ and $\beta \in [\frac{1}{2}, \frac{7}{8}[$, the system of root vectors of the operator $T(\varepsilon)$ is complete in $L^2(]-L, L[)$.* ◇

Proof. Let λ_n be the eigenvalue number n of T_0. We have

$$\lambda_{n+p}^{\frac{1}{4}} - \lambda_n^{\frac{1}{4}} = \kappa^{\frac{1}{4}}((n+p) - n) \geq 1, \quad \text{where } \alpha = \frac{3}{4} \text{ and } p \geq \frac{1}{\kappa^{\frac{1}{4}}}. \tag{4.5}$$

Then, in view of Corollary 3.1, Lemma 4.1 and Proposition 4.6 the system of root vectors of the operator $T(\varepsilon)$ is complete in $L^2(]-L, L[)$ for $|\varepsilon| < \frac{1}{\|K\|(1+\beta)}$. □

Theorem 4.2. *For small enough $|\varepsilon|$ and $\beta \in [\frac{1}{2}, \frac{7}{8}[$, the system of root vectors of the operator $T(\varepsilon)$ forms a basis with parentheses in $L^2(]-L, L[)$.* ◇

Proof. The result follows immediately from Corollary 3.2, Lemma 4.1, Proposition 4.6 and Eq. (4.5). □

Remark 4.3. Theorem 4.2 improves [11, *Theorem* 4.3]. Indeed, in [11] the authors proved that the system of root vectors of the operator $T(\varepsilon)$ forms an unconditional basis with parentheses in $L^2(]-L, L[)$ for $\beta \in [\frac{1}{2}, \frac{3}{4}]$, whereas in Theorem 4.2 we assure the existence of a basis with parentheses of root vectors for $\beta \in [\frac{1}{2}, \frac{7}{8}[$. ◇

References

[1] M. Aimar, A. IntissarA. Jeribi, *On an unconditional basis of generalized eigenvectors of the nonself-adjoint Gribov operator in Bargmann space.* J. Math. Anal. Appl. 231 (1999) 588–602.

[2] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform,* Comm. Pure Appl. Math. 14 (1961) 187–214.

[3] N. Ben Ali, A. Jeribi, *On the Riesz basis of a family of analytic operators in the sense of Kato and application to the problem of radiation of a vibrating structure in a light fluid,* J. Math. Anal. Appl. 320 (2006) 78–94.

[4] S. Charfi, A. Damergi, A. Jeribi, *On a Riesz basis of finite-dimensional invariant subspaces and application to Gribov operator in Bargmann space,* Linear Multilinear Algebra 61 (2013) 1577–1591.

[5] S. Charfi, A. Jeribi, I. Walha, *Riesz basis property of families of nonharmonic exponentials and application to a problem of a radiation of a vibrating structure in a light fluid,* Numer. Funct. Anal. Optim. 32 (4) (2011) 370–382.

[6] C. Clark, *On relatively bounded perturbations of ordinary differential operators,* Pacific J. Math. 25 (1968) 59–70.

[7] N. Dunford, J.T. Schwartz, *Linear Operators, Part III.* New York, Wiley-Interscience, 1971.

[8] H. Ellouz, I. Feki, A. Jeribi, *On a Riesz basis of exponentials related to the eigenvalues of an analytic operator and application to a non-selfadjoint problem deduced from a perturbation method for sound radiation,* J. Math. Phys. 54 (2013) 112101 pp 15.

[9] H. Ellouz, I. Feki, A. Jeribi, *On the asymptotic behavior of the eigenvalues of an analytic operator in the sense of Kato and applications,* Serdica Math. J. 45 (2019) no. 1, 55–88.

[10] H. Ellouz, I. Feki, A. Jeribi, *On a Riesz basis of exponentials related to a family of analytic operators and application,* J. Pseudo-Differ. Oper. Appl. 10 (2019) no. 4, 999–1014.

[11] I. Feki, A. Jeribi, R. Sfaxi, *On an unconditional basis of generalized eigenvectors of an analytic operator and application to a problem of radiation of a vibrating structure in a light fluid,* J. Math. Anal. Appl. 375 (2011) 261–269.

[12] I. Feki, A. Jeribi, R. Sfaxi, *On a Schauder basis related to the eigenvectors of a family of non-selfadjoint analytic operators and applications,* Anal. Math. Phys. 3 (2013) 311–331.

[13] I. Feki, A. Jeribi, R. Sfaxi, *On a Riesz basis of eigenvectors of a nonself-adjoint analytic operator and applications,* Linear Multilinear Algebra 62 (2014) 1049–1068.

[14] P.J.T. Filippi, O. Lagarrigue, P.O. Mattei, *Perturbation method for sound radiation by a vibrating plate in a light fluid: comparison with the exact solution,* J. Sound Vib. 177 (1994) 259–275.

[15] A. Intissar, *Analyse Fonctionnelle et Théorie Spectrale Pour les Opérateurs Compacts Non-Autoadjoints,* Editions CEPADUES, 1997.

[16] A. Intissar, A. Jeribi, I. Walha, *Riesz basis of exponential family for a hyperbolic system,* J. Appl. Anal. 25 (1) (2019) 13–23.

[17] A. Jeribi, *Spectral Theory and Applications of Linear Operators and Block Operator Matrices,* New-York, Springer-Verlag, 2015.

[18] A. Jeribi, *Denseness, Bases and Frames in Banach Spaces and Applications,* De Gruyter, Berlin, 2018.

[19] A. Jeribi, *Perturbation Theory for Linear Operators: Denseness and Bases with Applications.* Springer-Verlag (ISBN 978-981-16-2527-5), Singapore, 2021.

[20] A. Jeribi, A. Intissar, *On an Riesz basis of generalized eigenvectors of the nonselfadjoint problem deduced from a perturbation method for sound radiation by a vibrating plate in a light fluid,* J. Math. Anal. Appl. 292 (2004) 1–16.

[21] T. Kato, *Perturbation Theory for Linear Operators,* Berlin, Springer, 1980.

[22] A.S. Markus, *Introduction to the spectral theory of polynomial operator pencils,* Translations of Mathematical Monographs, 71, American Mathematical Society, Providence, RI, 1988.

[23] B.Sz. Nagy, *Perturbations des transformations linéaires fermées,* Acta Sci. Math. Szeged 14 (1951) 125-137.

[24] A.A. Shkalikov, *On the basis property of root vectors of a perturbed selfadjoint operator,* Tr. Mat. Inst. Steklova, 269, 290-303 (2010) (Russian); translation in Proc. Steklov Inst. Math. 269 (2010) 284–298.

[25] C. Wyss, *Riesz bases for p-subordinate perturbations of normal operators,* J. Funct. Anal. 258 (2010) 208–240.

Hanen Ellouz
email: ellouze.hanen@hotmail.fr
ORCID: 0000-0003-3801-7898
Département de Mathématiques
Université de Sfax
Faculté des sciences de Sfax
Route de soukra Km 3.5
B.P. 1171, 3000, Sfax
TUNISIE

Ines Feki
email: feki.ines@yahoo.fr
ORCID: 0000-0003-0046-8298
Département de Mathématiques
Université de Sfax
Faculté des sciences de Sfax
Route de soukra Km 3.5
B.P. 1171, 3000, Sfax
TUNISIE

Aref Jeribi
email: Aref.Jeribi@fss.rnu.tn
ORCID: 0000-0001-6715-5996
Département de Mathématiques
Université de Sfax
Faculté des sciences de Sfax
Route de soukra Km 3.5

B.P. 1171, 3000, Sfax
TUNISIE